TWISTED DUALITY AND POLYNOMIALS OF EMBEDDED GRAPHS

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ABSTRACT. We consider two operations on the edge of an embedded graph (or equivalently a ribbon graph): giving a half-twist to the edge and taking the partial dual with respect to the edge. These two operations give rise to an action of ${S_3}^{e(G)}$, the *ribbon group of* G, on G. We show that this ribbon group action gives a complete characterization of duality in that if G is any cellularly embedded graph with medial graph G_m, then the orbit of G under the group action is precisely the set of all graphs with medial graphs isomorphic (as abstract graphs) to G_m. We provide characterizations of special sets of twisted duals, such as the partial duals, of embedded graphs in terms of medial graphs and we show how different kinds of graph isomorphism give rise to these various notions of duality. We then show how the ribbon group action leads to a deeper understanding of the properties of, and relationships among, various graph polynomials such as the generalized transition polynomial, an extension of the Penrose polynomial to embedded graphs, and the topological Tutte polynomials of Las Vergnas and of Bollobás and Riordan.

1. INTRODUCTION

There are many investigations into planar and abstract graphs that have not yet been fully extended into the context of graphs embedded in closed surfaces (*i.e.* ribbon graphs). We explore two such extensions here. We broaden the notion of planar duality and its relation to medial graphs to *twisted duality* for embedded graphs. This allows us to give a complete characterization of all graphs with a given graph as their medial graph. We also adapt various graph polynomials, such as the Penrose polynomial of [71] and the generalized transition polynomial of [28], to embedded graphs, as Las Vernas [59] and Bollobäs and Riordan [9, 10] have done in extending the Tutte polynomial to encode topological information of ribbon graphs. Twisted duality then provides an especially apt tool for determining not only properties of these polynomials, but also how they relate to one another.

Our investigation into extensions of duality and the medial graph begins with the following property of planar graphs. Any planar graph G may be drawn in the plane, and both its planar dual, G^*, and its medial graph, G_m, constructed. Furthermore $(G^*)_m = G_m$. In fact, any abstract graph G has a 2-cell embedding in some surface, and from that embedding both a surface medial and a surface dual can be constructed, analogously to the planar case, and again the surface dual has the same medial graph as G. Now consider that medial graph, H, as an abstract graph. What precisely is the set of embedded graphs each with a medial graph isomorphic (as an abstract graph) to H? To answer this question we need a more flexible notion of duality than simply reversing the roles of the faces and the vertices. We need the partial duality, or duality with respect to an edge, of [17], but we also need the operation of giving an edge a half-twist. The result of applying combinations of these two operations to an embedded graph G gives a *twisted dual* of G. These two operations give rise to an action of ${S_3}^{e(G)}$ on G, which we call the *ribbon group action.* The ribbon group action, in the context of embedded graphs, gives a full understanding of the relations among embedded graphs and medial graphs in that if G is any cellularly embedded graph with medial graph G_m, then the orbit of G under the group action is precisely the set of all graphs with medial graphs isomorphic (as abstract graphs) to G_m.

The twisted duals that we introduce here include Chmutov's partial duals from [17] as a special case. Here we will show that partial duals are also admit a natural characterization in terms of medial graphs.

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We will see that the set of partial duals of G is exactly the set of embedded graphs whose medial graph is isomorphic to G_m when the medial graphs are considered as combinatorial maps.

These results allow us to take the point of view that concepts of equivalence of embedded graphs generate notions of duality. More specifically, that if $\sim$ denotes some notion of equivalence of embedded graphs, then $\{H \mid H_m \sim G_m\}$ is the set of dual graphs to G for some notion of duality. Here we will see that equivalent as embedded graphs generates Euler-Poincaré duality; equivalent as combinatorial maps generates partial duality; and equivalent as abstract graphs generates twisted duality. This provides a hierarchy and a framework for understanding generalizations of duality.

In another direction, most graph polynomials apply either to plane graphs (*e.g.* the Penrose polynomial of [71]) or to abstract graphs (*e.g.* the classical Tutte polynomial of [81, 82, 83]). Notable exceptions are the extension of the Tutte polynomial to embedded graphs by Las Vergnas in [59] and to ribbon graphs by Bollobás and Riordan in [9, 10]. We continue our investigation into the extensions of properties of planar and abstract graphs by adapting the generalized transition polynomial of [28] to embedded graphs. We use the embedding of a graph locally at the vertices to identify vertex state weights (weights for transition systems at the vertices) to achieve this. In this context the ribbon group action carries over to an action of the symmetric group on the vertex state weights, giving a twisted duality relation in terms of the weight systems for this polynomial.

The Penrose polynomial, defined implicitly for plane graphs by Penrose [71] in the context of tensor diagrams in physics, can be formulated in terms of vertex states of the medial graph. Thus, it can be given as an evaluation of a generalized transition polynomial with appropriately chosen vertex state weights (see Jaeger [45] and [27] for the planar case). The generalized transition polynomial, which applies to embedded graphs, then naturally extends the Penrose polynomial to embedded, *i.e.* ribbon, graphs. We then use twisted duality properties to strengthen and extend to embedded graphs many of the plane graph results for the Penrose polynomial from [2] and [45]. For example, we find an expression for the Penrose polynomial of a plane graph as a sum of chromatic polynomials, where the sum is indexed by a subgroup of the ribbon group. This generalizes a result of Aigner. A new equivalency statement for the Four Color Theorem follows from this.

With a different weight system, the generalized transition polynomial also agrees with the *topochromatic polynomial*, a shift of the topological Tutte polynomial of Bollobás and Riordan of [9, 10], for both the original and multivariate versions. This leads to twisted duality relations for the topochromatic polynomial as well. Thus, we conclude by applying twisted duality results in this area as well. In fact, much of our motivation for this study came from recent results relating knots and embedded graphs, and particularly the various recent connections between various realizations of knot polynomials as embedded graph polynomials ([21, 22, 17, 23, 33, 58, 65, 66, 67]). The topological Tutte polynomial and other related polynomials have recently arisen in physics [54, 55, 56, 39], and [55] uses the behavior of topological graph polynomials under the partial dual operation to calculate some parametric representation for commutative field theories. We give a fuller discussion of these interdisciplinary relations in Section 7.

Part of our interest in these problems derives from developing design strategies for self-assembling DNA nanostructures (see for example [48, 49, 73]). Because of the great promise of nanotechnology, especially for biomolecular computing, but also, for example, drug delivery and biosensors (see [87, 57]), recent research has focused on DNA self-assembly of nanoscale geometric constructs, notably graphs. Several different graphs have been constructed from self-assembling DNA strands, including cubes [16], truncated octahedra [89], rigid octahedra [74], and buckyballs [76]. An essential step in building a self-assembling DNA nano-construct, whether for biomolecular computing or physical applications, is designing the component molecules. The theory developed here for classifying graphs by their common medial graph suggests a possible efficient construction technique: first assemble a four-regular graph H, then use a biological process (such as an enzymatic action) to accomplish the splittings at the 4-armed branched junction molecules forming the 4-valent vertices of the graph. Double-stranded segments of DNA, with either an even or an odd number of twists, might then be introduced, binding to sites at the split vertices. These would form the edge segments to complete any of the various twisted duals that have H as a medial graph. The vertices of this new construct are the cycles of H. This allows one four-regular graph to act as a single molecular "template" for constructing a whole class of nanostructures. The polynomials considered here, which are generally

intractable to compute with current technology, might, in principle, be found by a biomolecular computing process of physical manipulations (splittings, cutting, and merging) of the molecules.

2. Embedded graphs

We use the term "embedded graph" loosely to mean any of three equivalent representations of graphs in surfaces. We may think of an embedded graph as any of:

(a) a cellularly embedded graph, that is, a graph embedded in a surface such that every face is a 2-cell;
(b) a ribbon graph, also known as a band decomposition;
(c) an arrow presentation.

In subsequent sections we will use these equivalent representations interchangeably, using whichever best facilitates the discussion at hand. We assume the reader is familiar with cellular embeddings of graphs, but we briefly review ribbon graphs, arrow presentations, and their equivalence (which is homeomorphism of the surface). We use standard notation: $V(G)$, $E(G)$, and $F(G)$ are the vertices, edges, and faces, respectively, of an embedded graph, while $v(G)$, $e(G)$, and $f(G)$ are the numbers of such. We will say that a loop in an embedded graph is *non-twisted* if a neighbourhood of it is orientable, and we say that the loop is *twisted* otherwise.

Definition 2.1. Following [10], a *ribbon graph* $G = (V(G), E(G))$ is a (possibly non-orientable) surface with boundary represented as the union of two sets of topological discs: a set $V(G)$ of *vertices*, and a set $E(G)$ of *edges* such that

(i) the vertices and edges intersect in disjoint line segments;
(ii) each such line segment lies on the boundary of precisely one vertex and precisely one edge;
(iii) every edge contains exactly two such line segments.

Two ribbon graphs are said to be *equivalent* or *isomorphic* if there is a homeomorphism between then that preserves the vertex-edge structure. (In particular, up to equivalence, the embedding of a ribbon graph in three-space is irrelevant.) We will say that a ribbon graph is *plane* if it is the neighbourhood of a plane graph, or equivalently, if the ribbon graph is a genus zero surface. A ribbon graph is *orientable* if it is orientable as a surface. Ribbon graphs are *band decompositions* (see Gross and Tucker [38]), with the 2-band interiors removed (the 2-bands correspond to faces). Note that in this context, $f(G)$ is just the number of boundary components of the surface with boundary comprising the ribbon graph G.

Definition 2.2. From Chmutov [17], an *arrow presentation* consists of a set of circles, each with a collection of disjoint, labelled arrows, called *marking arrows*, indicated along their perimeters. Each label appears on precisely two arrows.

Two arrow presentations are *equivalent* if one can be obtained from the other by reversing the direction of all of the marking arrows which belong to some subset of labels, or by changing the labelling set.

The equivalence of the three representations of embedded graphs is given rigorously in, for example [38]; however it is intuitively clear. If G is a cellularly embedded graph, a ribbon graph representation results from taking a small neighbourhood of the embedded graph G. This can be thought of as *cutting out* the ribbon graph from the surface. Neighbourhoods of vertices of the graph G form the vertices of a ribbon graph, and neighbourhoods of the edges of the embedded graph form the edges of the ribbon graph.

On the other hand, if G is a ribbon graph, we simply sew discs into each boundary component of the ribbon graph to get the desired surface. See Figure 1.

A ribbon graph can be obtained from an arrow presentation by viewing each circle as the boundary of a disc that becomes a vertex of the ribbon graph. Edges are then added to the vertex discs by taking a disc for each label of the marking arrows. Orient the edge discs arbitrarily and choose two non-intersecting arcs on the boundary of each of the edge discs. Orient these arcs according to the orientation of the edge disc. Finally, identify these two arcs with two marking arrows, both with the same label, aligning the direction of each arc consistently with the orientation of the marking arrow. See Figure 2.

Conversely, every ribbon graph gives rise to an arrow presentation. To describe a ribbon graph G as an arrow presentation, start by arbitrarily orienting and labelling each edge disc in $E(G)$. This induces an orientation on the boundary of each edge in $E(G)$. Now, on the arc where an edge disc intersects a vertex

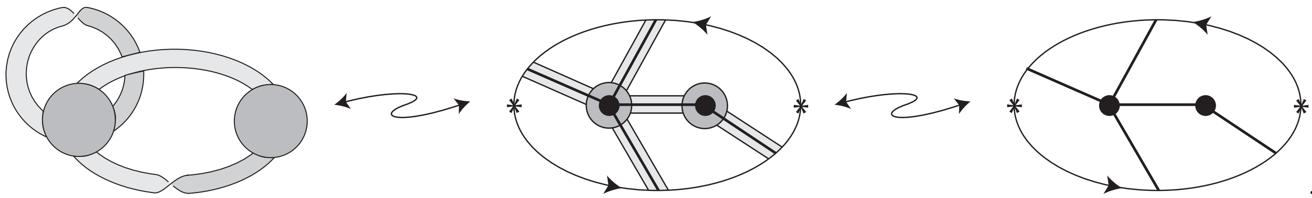


Figure 1. Equivalence of ribbon graphs and 2-cell embeddings.

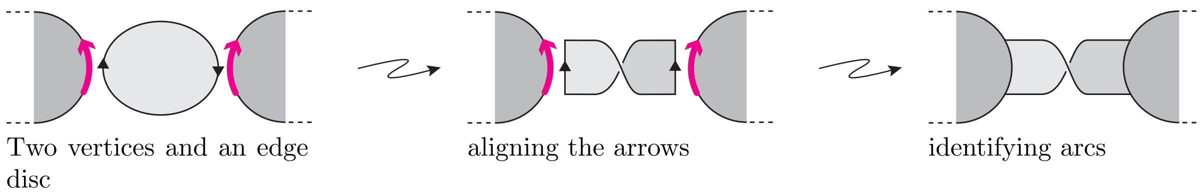


Figure 2. Constructing a ribbon graph from an arrow presentation.

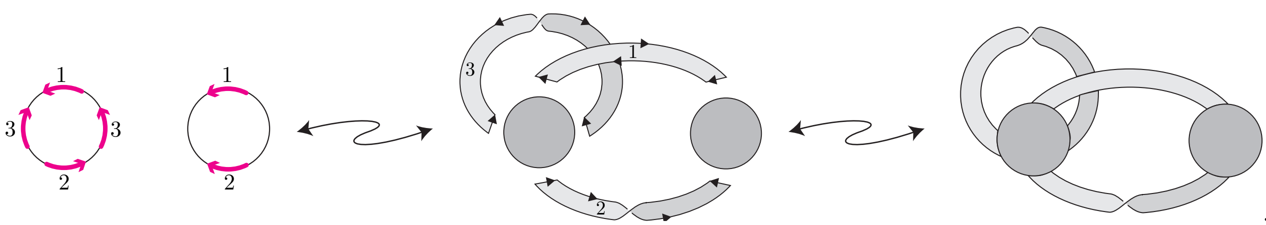


Figure 3. Equivalence of arrow presentations and ribbon graphs.

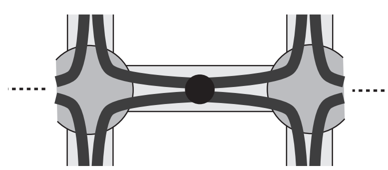
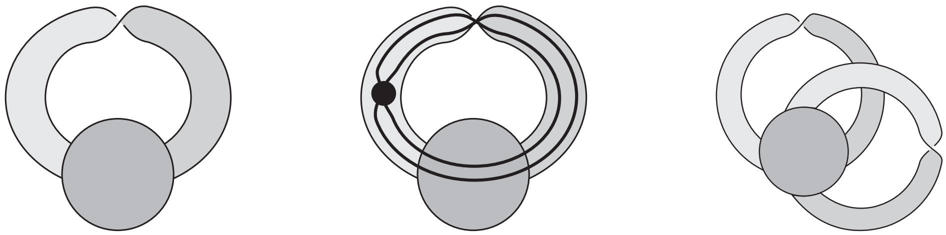


Figure 4. The formation of a medial ribbon graph.

disc, place a marked arrow on the vertex disc, labelling the arrow with the label of the edge it meets and directing it consistently with the orientation of the edge disc boundary. The boundaries of the vertex set marked with these labelled arrows give the arrow marked circles of an arrow presentation. See Figure 3.

The three ways of representing embedded graphs are equivalent, and furthermore, two graphs are equivalent as ribbon graphs if and only if they are equivalent as cellularly embedded graphs if and only if they are equivalent as arrow presentation. When two embedded graphs are equivalent in this sense, we will often say that they are *equivalent as embedded graphs* or are *isomorphic*, to distinguish this kind equivalence from other graph equivalences that arise in this paper. In particular, we will be interested in how different notions of equivalence generate extensions of the concept of the dual of a graph.

Two other natural notions of equivalence are important here. There is a natural forgetful functor from the category of embedded graphs to the category of graphs which forgets all of the information concerning the embedding. Thus every embedded graph gives rise to an (abstract) graph by forgetting the embedding and retaining only the vertices and their adjacency information. A cellularly embedded graph G consists of a graph G' and a cellular embedding ψ of G' into a surface. We say that (the abstract graph) G' is



A ribbon graph G. The medial graph G_m drawn inside G. G_m presented as a ribbon graph.

Figure 5. An example of a medial ribbon graph.

the *underlying graph* of (the embedded graph) G. We will say that two embedded graphs are *equivalent as abstract graphs* if their underlying graphs are isomorphic (or equivalent) as abstract graphs. We will usual denote isomorphic as abstract graphs by $G \cong F$, reserving $G = F$ for when G and F are isomorphic as embedded graphs.

We will also be interested in equivalence as combinatorial maps. This equivalence is stronger than equivalence as abstract graphs, but weaker than equivalence as embedded graphs. A *combinatorial map* G is a graph G' equipped with a cyclic order $\mathcal{O}$ of the incident half-edges at each vertex. Combinatorial maps describe graphs embedded in orientable surfaces (see [38] for example). Every embedded graph G has an *underlying combinatorial map* which is obtained by forgetting the twists as follows. Working with the realization of G as a ribbon graph, simply remove then half-twists from the edges of G to obtain an orientable ribbon graph G'. G' describes a graph embedded in an orientable surface which gives rise to a combinatorial map in the usual way: orienting G' induces a cyclic order of the incident half-edges at each vertex. The combinatorial map then is obtained as the underlying graph equipped with the induced cyclic order at the vertices. Alternatively, given a arrow representation of an embedded graph G, one can obtain the underlying combinatorial map by 'forgetting' the directions on the arrows and retaining only their cyclic order about the vertex disc, or simply switching directions as needed so that all arrows point in a counterclockwise direction. Both of these approaches give a natural forgetful functor from the category of embedded graphs to the category of combinatorial maps. We will say that two embedded graphs are *equivalent as combinatorial maps* if their underlying combinatorial maps are equivalent. (See [38], for example, for a definition of equivalence of combinatorial maps.)

Medial graphs also play a central role throughout the rest of this paper. If G is cellularly embedded, we constructed its medial graph G_m exactly as in the plane case, by placing a vertex of degree 4 on each edge, and then drawing the edges of the medial graph by following the face boundaries of G. There is a natural embedding of the medial graph, viewed as a ribbon graph, into G viewed as a ribbon graph. This results from drawing the edges of G_m very close to the the edges of G in the surface, and taking a smaller neighbourhood of G_m than the neighbourhood of G when cutting out the ribbon graphs from the surface. See Figure 4. An example of a medial ribbon graph is given in Figure 5. Consistent with this definition is that the medial graph of an isolated vertex is again an isolated vertex, and we adopt this convention.

Ribbon graphs provide a particularly apt model for our motivating application of DNA nanoconstructs. The techniques of [1, 89, 16], and in essence [74], use single strands of DNA to trace out each edge of a graph, once in each direction, binding to themselves along segments of Watson-Crick complements to form double-helixes along the edges and stable branched junctions at the vertices. The sequences of nucleotides are carefully specified so that only the two sides of an edge are complements of, and hence bind to, each other. A strand may thus be interpreted as a facial walk of a graph embedded in a surface (see [47] for the topological formalism), or equivalently as the boundary of a ribbon graph, with orientability determined by the parity of the number of twists in the double strands of DNA forming the edges.

3. Twisted duality and the ribbon group action

Duality in the plane is constrained by restricting the result of taking the dual to be again a plane graph. Working with embedded graphs enables greater flexibility. This has been realized with the generalized duality

FIGURE 6. τ and δ with arrow presentations.

of Chmutov [17], which involves dualizing with respect to individual edges, and hence moving out of the class of plane graphs. Here we extend this idea further, allowing two operations on the edges of an embedded graph G: not only taking the dual with respect to an edge, but also giving a half-twist to an edge. These two operations give rise to a group action of ${S_3}^{e(G)}$ on G, which we call the *ribbon group action*. The ribbon group action is the foundation of many of the results in the later sections of this paper.

3.1. **The ribbon group action.** As often is the case, we begin with graphs equipped with a linear ordering on their edges, and then show that our constructions are independent of these orderings. We first give the twist and dual operations with respect to single distinguished edges. We define the *ribbon group action* of $\mathfrak{G}^{e(G)}$ on graphs with a linear ordering on their edges, where $\mathfrak{G} \cong S_3$. We then provide a more efficient notation which is independent of the ordering of the edges and also establish some elementary properties of the ribbon group action.

We write $\mathcal{G}$ for the set of all embedded graphs considered up to homeomorphism (we will identify G with its homeomorphism class), and $\mathcal{G}_{(n)} \subseteq \mathcal{G}$ for those with n edges. Similarly, we write

$$\mathcal{G}_{or} = \{(G,\ell) | G \in \mathcal{G} \text{ and } l \text{ is a linear order of the edges}\}$$

for the set of embedded graphs with ordered edges, and

$$\mathcal{G}_{or(n)} = \big\{(G,\ell) | G \in \mathcal{G}_{(n)} \text{ and } \ell \text{ is a linear order of the edges}\big\}$$

for those with n edges.

Definition 3.1. Let $(G,\ell) \in \mathcal{G}_{or}$ and suppose e_i is the i^{th} edge in the ordering l. Also, suppose G is given in term of its arrow presentation, so e_i is a label of a pair of arrows.

The *half-twist of the i^{th} edge* is $(\tau,i)(G,\ell) = (H,\ell)$ where H is obtained from G by reversing the direction of exactly one of the e_i-labelled arrows of the arrow presentation, as in Figure 6. Most importantly, H inherits its edge order ℓ in the natural way from G.

The *dual with respect to the i^{th} edge* is $(\delta,i)(G,\ell) = (H,\ell)$, where H is obtained from G as follows. Suppose A and B are the two arrows labelled e_i in the arrow presentation of G. Draw a line segment with an arrow from the the head of A to the tail of B, and a line segment with an arrow from the head of B to the tail of A. Label both of these arrows e_i, and delete A and B with the arcs containing them. The line segments with their arrows become arcs of a new circle in the arrow presentation of H. As with the twist, H here inherits its edge order l from G. See Figure 6.

We make the simple but important observation that these operations applied to *different* edges commute.

Proposition 3.2. *If $i \neq j$ and $\xi,\zeta \in \{\tau,\delta\}$, then $(\xi,i)((\zeta,j)(G,\ell)) = (\zeta,j)((\xi,i)(G,\ell))$.*

However, (τ,i) and (δ,i) do *not* commute when applied repeatedly to the same edge, and in fact we will see they induce a group action of S_3 on that edge.

We use the following notation to denote compositions applied to the same edge:

$$(\xi\zeta,i)(G,\ell) := (\xi,i)((\zeta,i)(G,\ell)),$$

where $\xi,\zeta \in \{\tau,\delta\}$. We also define $(1,i)(G,\ell) := (G,\ell)$. Thus, we can consider the action of (ξ,i) on (G,ℓ) where ξ is a word in $\{\tau,\delta\}$.

Lemma 3.3. *If $(G,\ell) \in \mathcal{G}_{or}$ then, for each fixed i,*

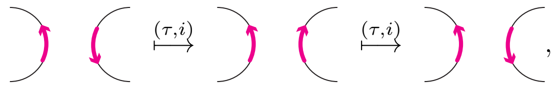
$$(\tau^2,i)(G,\ell) = (\delta^2,i)(G,\ell) = ((\tau\delta)^3,i)(G,\ell) = 1(G,\ell).$$

Therefore, given a fixed i, there is an action of the symmetric group S_3, with the presentation

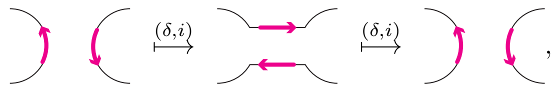
$$S_3 \cong \mathfrak{G} := \langle \delta,\tau \mid \delta^2,\tau^2,(\tau\delta)^3\rangle,$$

on $\mathcal{G}_{or}$.

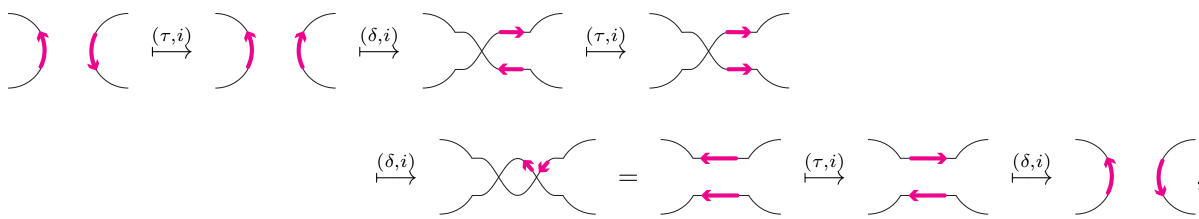
Proof. The following calculations verify that $(\delta^2, i)(G, \ell) = (\tau^2, i)(G, \ell) = ((\tau\delta)^3, i)(G, \ell) = 1(G, \ell)$. Since



then $(\tau^2, i) = 1$. Also



giving the identity $(\delta^2, i) = 1$. Finally,



and thus $((\tau\delta)^3, i) = 1$, completing the proof of the lemma. $\square$

The action in Lemma 3.3 for fixed i now extends to a group action of ${S_3}^{e(G)}$ on $\mathcal{G}_{or(n)}$.

Definition 3.4. We call ${S_3}^n \cong \mathfrak{G}^n$ the *ribbon group for n edges* and define the *ribbon group action* of the ribbon group on $\mathcal{G}_{or(n)}$ by:

$$\begin{aligned}(\xi_1, \xi_2, \xi_3, \dots, \xi_n)(G, \ell) \quad &= (\xi_n, n)((\xi_{n-1}, n-1) \dots ((\xi_2, 2)((\xi_1, 1)(G, \ell))) \dots) \\ &= ((\xi_n, n) \circ (\xi_{n-1}, n-1) \circ \dots \circ (\xi_2, 2) \circ (\xi_1, 1))(G, \ell),\end{aligned}$$

where $\xi_i \in \mathfrak{G}$ for all i.

With Definition 3.4, if $(G, \ell) \in \mathcal{G}_{or(n)}$, then we can view (τ, i) as an element of $\mathfrak{G}^n$ of the form $(1, \dots, \tau, \dots, 1)$, with τ in the i^{th} coordinate, and similarly for δ and the other elements of $\mathfrak{G}$.

3.2. **Twisted duals.** We now define twisted duality for graphs without any edge ordering. Our final definition of a twisted dual of G will be of the form

$$G^{\prod \xi_i(A_i)},$$

where the A_i's partition the edge set, and the ξ_i's are in $\mathfrak{G}$. However, we need some preliminary definitions to make sense of this expression.

We begin with an obvious proposition.

Proposition 3.5. *If $(G, \ell) \in \mathcal{G}_{or(n)}$, $\boldsymbol{\zeta} \in \mathfrak{G}^n$, and $\sigma \in S_n$, then $\boldsymbol{\zeta}(G, \ell) = \sigma(\boldsymbol{\zeta})(G, \sigma(\ell))$, where σ acts on $\boldsymbol{\zeta}$ by permuting the order of the elements of the n-tuple.*

Proposition 3.5 assures that $G^{\prod \xi(A_i)}$ as given below is well-defined, that is, it is independent of the edge ordering ℓ.

Definition 3.6. Suppose $G \in \mathcal{G}_{(n)}$, $A, B \subseteq E(G)$ and $\xi, \zeta \in \mathfrak{G}$. Define $G^{\xi(A)}$ as follows. Let ℓ be an arbitrary ordering $(e_1, \dots, e_n)$ of the edges of G, and define $\boldsymbol{\xi}_A := (\epsilon_1, \dots, \epsilon_n) \in \mathfrak{G}^n$, where $\epsilon_i = \xi$ if $e_i \in A$ and $\epsilon_i = 1$ else. Then,

$$G^{\xi(A)} := \boldsymbol{\xi}_A(G, \ell).$$

Moreover, we establish the following notational conventions:

$$G^{\xi(A)\zeta(B)} := (G^{\xi(A)})^{\zeta(B)}, \text{ and}$$

$$G^{\xi\zeta(A)} := G^{\zeta(A)\xi(A)}.$$

Proposition 3.7. *If, for all i, we have $\zeta_i \in \mathfrak{G}$ and $B_i \subseteq E(G)$, then any expression of the form*

$$G^{\prod \zeta_i(B_i)},$$

(where the product is not *necessarily commutative) is equal to*

$$G^{\prod_{i=1}^{6} \xi_i(A_i)},$$

where the product is *commutative, and where the $A_i \subseteq E(G)$ are pairwise disjoint with $\cup_i A_i = E(G)$, and where $\xi_1 = 1, \xi_2 = \tau, \xi_3 = \delta, \xi_4 = \tau\delta, \xi_5 = \delta\tau, \xi_6 = \tau\delta\tau \in \mathfrak{G}$.*

Proposition 3.7 follows from repeated applications of Definition 3.6, and the fact that the product commutes follows from the fact that the A_i's are disjoint and Proposition 3.2. Hereafter, we will customarily omit any factors of the form $1(A_1)$ or $\xi_i(\emptyset)$ in these expressions. Also, if A_i is given explicitly by a list of edges, to simplify notation, we will omit the set brackets, for example, writing $\tau(e, f)$ for $\tau(\{e, f\})$.

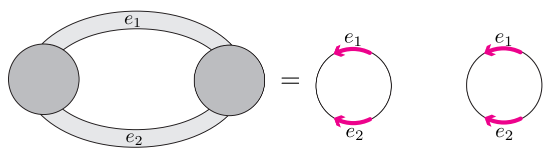
As an example of Proposition 3.7, if G is an embedded graph with edges d, e, f, g, h, then

$$G^{\tau(d,e,f)\delta(e,f,g)} = (G^{\tau(d,e,f)})^{\delta(e,f,g)} = G^{\tau(d)(\tau(e,f)\delta(e,f))(\delta(g))} = G^{\tau(d)\delta\tau(e,f)(\delta(g))} = G^{\tau(d)\delta(g)\delta\tau(e,f)}.$$

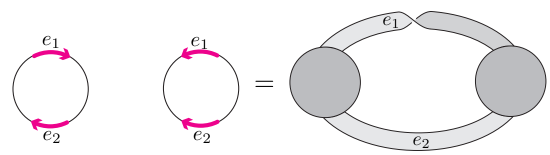
The edge h is unaffected.

Geometrically, these maps act on ribbon graphs in the following way: τ adds a half-twist to the edge e, and δ forms the partial dual (introduced by Chmutov [17] and further studied in [67, 68]), at the edge e. Products of τ and δ are applied to the edge successively. An illustration of the actions of τ and δ on a ribbon graph is given in Example 3.8.

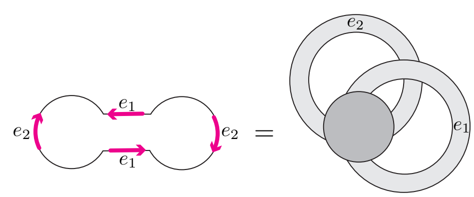
Example 3.8. If G is an embedded graph with $E(G) = \{e_1, e_2\}$, with the order (e_1, e_2), represented as a ribbon graph and an arrow presentation as shown below,

$G =$ 

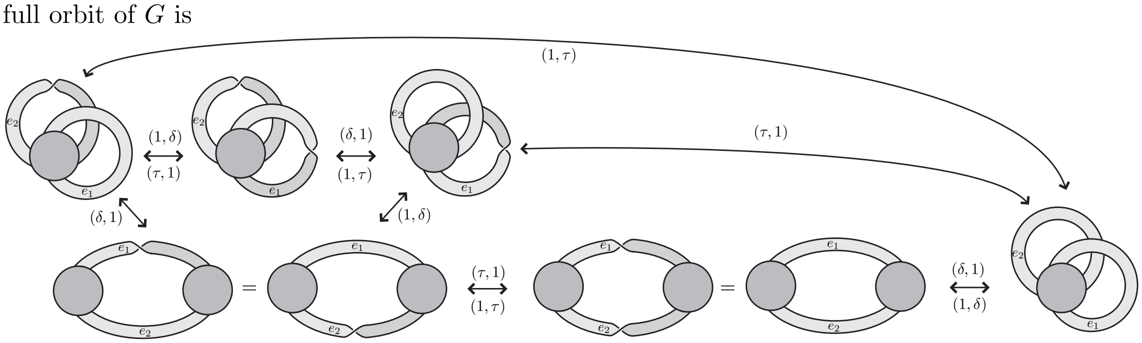
then we have

$(\tau, 1)(G) = G^{\tau(e_1)} =$ 

and

$(\delta, 1)(G) = G^{\delta(e_1)} =$ 

The full orbit of G is



We now have the following definition of a *twisted dual of an embedded graph*.

Definition 3.9. If G is an embedded graph, then H is a *twisted dual* of G if it can be written in the form

$$H = G^{\prod_{i=1}^{6} \xi_i(A_i)},$$

where the A_i's partition $E(G)$, and the ξ_i's are the six elements of $\mathfrak{G}$.

Equivalently, if l is an arbitrary ordering of the edges of G, then H is a *twisted dual* of G if $\boldsymbol{\xi}(G, \ell) = (H, \ell)$ for some $\boldsymbol{\xi} \in \mathfrak{G}^{e(G)}$, *i.e.* H is a twisted dual of G if (H, ℓ) is in the orbit of (G, ℓ) under the ribbon group action. Note that twisted duality is symmetric, so we may speak of H and G as being twisted duals of one another.

The linear ordering on the edges is necessary to define a group action (since there is no "universal edge set" for graphs as there is, say, a universal point set for a class of matroids), but not necessary to the geometric construction of twisted duals. Thus we will use Definition 3.4 when we wish to emphasize the group action, and Definition 3.9 to emphasize the geometry. However, since moving between the two viewpoints is so natural, our language may not always preserve the distinction, as for example, we may speak of $\mathcal{G}^n$ "acting" on an unordered graph.

One of our main interests in this paper is in the orbits $Orb(G, \ell) := \mathfrak{G}^{e(G)}(G, \ell) = \{\boldsymbol{\xi}(G, \ell) \mid \boldsymbol{\xi} \in \mathfrak{G}^{e(G)}\}$ of the group action. With slight abuse of terminology, we define

$$Orb(G) := \{H : (H, \ell) \in Orb(G, \ell) \text{ for some edge order } \ell\}.$$

3.3. **Some notable subgroups of the ribbon group.** Twisted duality generalizes the notion of the partial dual of a ribbon graph. Partial duality is intimately related with knot theory, and can be used to represent link diagrams ([23, 66, 22, 17]) without the need for using edge weights to record the under/over crossings of the link. The more general construction of twisted duality provides new connections between knot theory and graph theory. For example, Chmutov [18]observed that Seifert surfaces for a link can be found among the twisted duals of any one of its Tait graphs. We also note that some classic results on duality extend to partial and twisted duality, for example see [68], where the second author gave a characterization of partially dual graphs in terms of edge bijections between graphs in the spirit of Edmonds [25], and of course, the results presented later in this paper.

We take a moment now to place prior work on partial duality in the context of twisted duality.

Definition 3.10. Let $\mathfrak{G}_{pd} = \langle \delta \mid \delta^2 \rangle$ be the subgroup of $\mathfrak{G}$ generated by δ. Then two twisted duals H and G are said to be *partial duals* if $(H, \ell) = \boldsymbol{\xi}(G, \ell)$ for some $\boldsymbol{\xi} \in \mathfrak{G}_{pd}^{e(G)}$ and some edge ordering ℓ.

The following fact relating twisted duality and Euler-Poincaré duality will be important later. Recall that the Euler-Poincaré dual G^* of an cellularly embedded graph G is constructed exactly as in the plane case by placing a vertex in each face, and connecting two of these vertices with an edge whenever their faces share an edge on their boundaries. In the context of ribbon graphs, G^* is constructed by regarding the ribbon graph G as a punctured surface, filling in the punctures using a set of discs denoted $V(G^*)$, then removing the original vertex set $V(G)$ (so $G^* = (V(G^*), E(G))$).

Proposition 3.11 (Chmutov [17])**.** *If G is an embedded graph, then*

$$G^* = G^{\delta(E(G))}.$$

Chmutov further observed in [17] that partial duality can be regarded as an action of $\mathbb{Z}_2^{E(G)}$ on a ribbon graph G.

A second subgroup of the ribbon group of interest here is the subgroup generated by twists: $\mathfrak{G}_{tw} = \langle \tau \mid \tau^2 \rangle \leq \mathfrak{G}$. We will say that an embedded graph G is a *twist* of H if $(G, \ell) = \boldsymbol{\xi}(H, \ell)$ for some $\boldsymbol{\xi} \in \mathfrak{G}_{tw}^{e(G)}$ and some edge ordering ℓ. We can use the orbit of the action of $\mathfrak{G}_{tw}$ to describe a certain set of embeddings of a combinatorial map. By a *locally admissible embedding* of a combinatorial map M, we mean a cellular embedding of M into some surface such that the cyclic order at the vertices of M is preserved with respect to one of the local orientation of a neighbourhood of the image of each vertex. For example, the 3-regular plane graph with two vertices and the 3-regular, two vertex graph embedded in the torus are both locally admissible embeddings of the same combinatorial map, in addition, this map will have locally admissible embeddings on non-orientable surfaces such as the real projective plane. The following proposition is clear from the definition of the underlying combinatorial map of an embedded graph G.

Proposition 3.12. *Let G be an embedded graph with underlying combinatorial map M. Then the set of twists of G is equal to the set of locally admissible embeddings of M, that is*

$$\{\boldsymbol{\xi}(G,\ell) \mid \boldsymbol{\xi} \in \mathfrak{G}^{e(G)}_{tw}\} = \{G^{\tau(A)} \mid A \subseteq E(G)\} = \{M^{\tau(A)} \mid A \subseteq E(G) = E(M)\} = \{\textit{locally admissible embeddings of } M\},$$

where ℓ is any linear order of $E(G)$.

3.4. **Some properties of the ribbon group action.** In general, the group action $\mathfrak{G}^n$ on $\mathcal{G}_{or(n)}$ is faithful and transitive, but is not free and has no fixed points. However, it turns out that stronger, label-independent analogues of these properties hold for twisted duality. We will state and prove these stronger analogous properties in Proposition 3.13 and then state and prove the ordered versions in the language of group actions in Corollary 3.14.

Proposition 3.13.

(i) Let $\boldsymbol{\xi} \in \mathfrak{G}^n$, then $\boldsymbol{\xi}(G,\ell) \in \{(G,\pi(\ell)) \mid \pi \in S_n\}$ for all $G \in \mathcal{G}_n$ if and only if $\boldsymbol{\xi} = \mathbf{1}$.
(ii) For all $G \in \mathcal{G}_n$ there is some $\boldsymbol{\xi} \in \mathfrak{G}^n$ such that $\boldsymbol{\xi}(G,\ell) \notin \{(G,\pi(\ell) \mid \pi \in S_n\}$.
(iii) For all $G \in \mathcal{G}_n$, there exists an $G' \in \mathcal{G}_n$ such that G' is not a twisted dual of G, if and only if $n > 1$

Proof. (i) The sufficiency is easily verified by calculation.

To prove the necessity, for each $\boldsymbol{\xi} \neq \mathbf{1}$ we find $(G,\ell) \in \mathcal{G}_{or(n)}$ with the property that, if $\boldsymbol{\xi}(G,\ell) = (H,\ell)$, then $G \neq H$ as embedded graphs, and hence $(H,\ell) \neq (G,\pi(\ell))$ for any ordering $\pi(\ell)$.

Given $\boldsymbol{\xi} \neq \mathbf{1}$, so that $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$ has $\xi_i \neq 1$ for some i, we consider $(B,\ell), (B',\ell) \in \mathcal{G}_{or(n)}$, where B is the connected plane bouquet, which consists of n loops at a single vertex, and $(\tau, i)(B,\ell) = (B',\ell)$.

If $\xi_i = \delta$ or $\tau\delta$, then $\boldsymbol{\xi}(B,\ell)$ has at least one more vertex than (B,ℓ) (since $(\xi_i, i)(B,\ell)$ has two vertices), so $\xi(B,\ell) \neq (B,\pi(\ell))$ for any permutation π. If $\xi_i = \tau$ then $\boldsymbol{\xi}(B,\ell)$ is non-orientable (as $(\xi_i, i)(B,\ell)$ is non-orientable), so again $\xi(B,\ell) \neq (B,\pi(\ell))$ for any permutation π. If $\xi_i = \delta\tau$ or $\tau\delta\tau$, then $\boldsymbol{\xi}(B',\ell)$ has at least one more vertex than (B',ℓ) (since $(\xi_i, i)(B',\ell)$ has two vertices). An all of these cases the $\boldsymbol{\xi}$ changes the underlying graph as required.

(ii) Let $(G,\ell) \in \mathcal{G}_{\mathrm{or}(n)}$, then G either contains a cycle or does not contain a cycle. If G contains a cycle then there exists a set of edges A of G such that adding a half-twist to each of the edges in A will change the orientability of G. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$, where $\xi_i = \tau$ if $e_i \in A$ and 0 otherwise. It then follows that exactly one of (G,ℓ) and $\boldsymbol{\xi}(G,\ell)$ is orientable.

On the other hand, if G does not contain a cycle, then taking the dual at any edge e of G will result in a ribbon graph a ribbon graph containing a cycle. In this case, if we let $\boldsymbol{\xi} = ((\delta,1),(1,2),\dots(1,n))$, we have that exactly one of (G,ℓ) and $\boldsymbol{\xi}(G,\ell)$ contains a cycle.

In either case there exists some $\boldsymbol{\xi} \in \mathfrak{G}^n$ with the property that the ribbon graphs (without an edge order) in (G,ℓ) and $\boldsymbol{\xi}(G,\ell)$ are distinct, and therefore $(G,\ell) \neq \boldsymbol{\xi}(G,\pi(\ell))$, for any edge order $\pi(\ell)$.

(iii) Sufficiency is easily checked by calculation. To prove the necessity, assume that $n > 1$. We need to show that for any graph $G \in \mathcal{G}_n$ there is some graph $H \in \mathcal{G}_n$ which is not a twisted dual of G.

We will prove the result by showing that there exists a set $\mathcal{S}$ of ribbon graphs in $\mathcal{G}_n$ that is closed under taking the twisted duals and that has the additional property that every orientable ribbon graph in the set is plane. Thus, any graph $G \in \mathcal{G}_n - \mathcal{S}$ is not a twisted dual of any graph in $\mathcal{S}$ and vice versa. However, for all $n > 1$, there is an orientable non- plane ribbon graph, so $\mathcal{G}_n - \mathcal{S}$ is not empty.

Our desired set $\mathcal{S}$ has the property that $G \in \mathcal{S}$ if and only if G is plane and has a distinguished vertex v such that every edge in G is either a loop incident with v or the edge is a bridge incident with v and a 1-valent vertex; or G is a twist of such a ribbon graph. Notice that every orientable ribbon graph in $\mathcal{S}$ is plane.

To show that $\mathcal{S}$ is closed under the operation of twisted duality it is enough to show that for each $G \in \mathcal{S}$, $e \in E(G)$ and $\xi \in \mathfrak{G}$, we have $G^{\xi(e)} \in \mathcal{S}$. To see that this is indeed the case let e be an edge of some $G \in \mathcal{S}$. Then e is either a non-twisted loop, a twisted loop, or a bridge, and $\xi = 1, \tau, \delta, \tau\delta, \delta\tau$ or $\tau\delta\tau$. If e is a non-twisted loop then the edge corresponding to e in $G^{1(e)}$ and $G^{\tau\delta\tau(e)}$ is a non-twisted loop; in $G^{\delta(e)}$ and $G^{\tau\delta(e)}$ it is a bridge; and in $G^{\tau(e)}$ and $G^{\delta\tau(e)}$ it is a twisted loop. If e is a twisted loop then the edge corresponding to e in $G^{\tau(e)}$ and $G^{\tau\delta(e)}$ is a non-twisted loop; in $G^{\delta\tau(e)}$ and $G^{\tau\delta\tau(e)}$

is a bridge; and in $G^{1(e)}$ and $G^{\delta(e)}$ is a twisted loop. If e is a bridge then the edge corresponding to e in $G^{\delta(e)}$ and $G^{\delta\tau(e)}$ is a non-twisted loop; in $G^{\delta\tau(e)}$ and $G^{\tau\delta\tau(e)}$ is a bridge; and in $G^{1(e)}$ and $G^{\delta(e)}$ is a twisted loop. In all off these cases the resulting twisted dual $G^{\xi(e)}$ is in $\mathcal{S}$ and the result then follows.

□

Corollary 3.14. *The action of $\mathfrak{G}^n$ on $\mathcal{G}_{or(n)}$ is*

(i) faithful;
(ii) has no fixed points;
(iii) transitive if and only if $n > 1$;
(iv) not free;

Proof. The first three properties follow easily from Proposition 3.13.

To show that the action is not free we need to show that there is some $(G, \ell) \in \mathcal{G}_n$, such that $\xi(G, \ell) = (G, \ell)$ for some non- trivial $\boldsymbol{\xi} \in \mathfrak{G}^n$. This exists since $\prod_{i=1}^{n}(\tau, i)(K_{1,n}, \ell) = (K_{1,n}, \ell)$, for any edge order ℓ of the complete bipartite ribbon graph $K_{1,n}$. □

Remark 3.15. In this paper we investigate and characterize the orbits of the ribbon group action. In addition to the orbits, the stabilizer subgroup of $\mathfrak{G}^{e(G)}$, like the automorphism group, is an invariant of G, and thus warrants further study. Another interesting question that we do not address here, is how ribbon graph theoretic properties, such as the genus or number of vertices, vary over the elements in an orbit.

4. Medial graphs and the ribbon group action

Via the ribbon group action, twisted duality gives the full story of the interplay among a graph, its medial graph, and its various twisted duals. This allows us to answer the question we originally posed: Given any 4-regular graph F, what precisely is the set of embedded graphs that have medial graphs isomorphic to F as abstract graphs? We are simultaneously able to provide a classification of all the twisted duals of an embedded graph G via its medial graph, and thus characterize $Orb(G)$. These relations among twisted duals, embedded medial graphs, and cycle family graphs are higher genus generalizations of the classic results relating plane duals, medial graphs, and Tait graphs.

By way of motivation, we begin by reviewing some basic properties of plane medial graphs. If F is a connected 4-regular plane graph, then F is face two-colourable (see *e.g.* Fleischner [35]), and we call this a *checkerboard colouring*, and use the colours black and white. The *blackface graph*, F_{bl}, of F is the plane graph constructed by placing one vertex in each black face and adding an edge between two of these vertices whenever the corresponding regions meet at a vertex of F. The *whiteface graph*, F_{wh}, is constructed analogously by placing the vertices in the white faces. Borrowing terminology from knot theory, we refer to F_{bl} and F_{wh} as the *Tait graphs* of F.

There are two key properties of Tait graphs. The first key property is duality: $(F_{bl})^* = F_{wh}$ and vice versa, where the asterisk indicates planar duality. Thus, if G is any plane graph, and we give G_m the *canonical checkerboard colouring*, *i.e.* where the black faces contain the vertices of G, then

$$(G_m)_{bl} = G, \text{ and } (G_m)_{wh} = G^*. \tag{4.1}$$

Secondly, the medial graph of a Tait graph is just the original graph, *i.e.*

$$(F_{wh})_m = (F_{bl})_m = F. \tag{4.2}$$

Moreover, $\{F_{wh}, F_{bl}\}$ is exactly the set of plane graphs whose medial graph is F.

With this, we can think of the Tait graphs loosely as "orbits" of size two under the operation of planar duality, and everything in this "orbit" shares the same medial graph. In Section 4.3 we will see how this point of view is fully realized by embedded graphs.

In the special case that a 4-regular embedded graph F (thought of as being cellularly embedded in a surface) is checkerboard colourable, then we can construct the Tait graphs just as in the plane case, and the same properties described above will still hold. In particular, there are the following two well-known results.

Proposition 4.1. *If G is any embedded graph, thought of as being cellularly embedded in a surface, and we canonically checkerboard colour the embedded medial graph G_m, then $(G_m)_{bl} = G$ and $(G_m)_{wh} = G^* = G^{\delta(E(G))}$, the Euler-Poincaré dual of G in the surface.*

Proposition 4.2. *Let F be a 4-regular, cellularly embedded graph. Then*

(i) if F is checkerboard colourable, then $\{F_{bl}, F_{wh}\}$ is the complete set of cellularly embedded graph with embedded medial graph equivalent to F;
(ii) if F is not checkerboard colourable then F is not the embedded medial graph of any embedded graph.

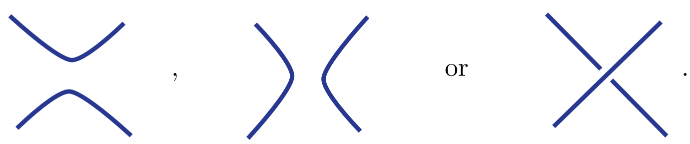
The two propositions above provide a complete characterization of the embedded graphs whose medial graph is equivalent to a given 4-regular, embedded graph F. Moreover, the propositions tell us that all of the embedded graphs with this property are duals.

A natural question then arises: Given any 4-regular graph F, which embedded graphs have medial graphs isomorphic to F as abstract graphs? In this section we answer this question, giving a complete characterization of the set of embedded graphs with this property. To do this we introduce the concept of the cycle family graphs of a 4-regular, embedded graph F. The cycle family graphs do not rely on the checkerboard colourability of an embedded graph and every 4-regular, embedded graph will admit a set of cycle family graphs. We will prove that,

(i) $G_m \cong H_m$ as abstract graphs if and only if G and H are twisted duals (compare Equation (4.1) and Proposition 4.1 to Theorem 4.10).
(ii) $G_m \cong F$ as abstract graphs for a given 4-regular, embedded graph F if and only if G is a cycle family graph of F (compare Equation (4.2) and Proposition 4.2 to Theorem 4.12);

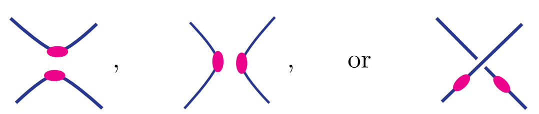
Thus the relationships between cycle family graphs and twisted duals fully extend the classic relations between the Tait graphs and duality.

4.1. **Cycle family graphs.** Let F be a 4-regular embedded graph thought of as a 2-cell embedding. A *vertex state* of $v \in V(F)$ is a choice of one of the following reconfigurations in a neighborhood of the vertex v:



The configurations replace a small neighbourhood of the vertex v. We will refer to the first two of the these vertex states as *splits* and the third as a *crossing*. Vertex states are sometimes called transitions or transition systems, but here again we choose terminology that is closer to that of knot theory.

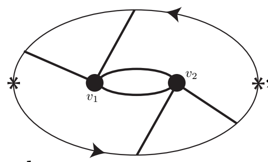
An *arrow marked vertex state* of v consists of a vertex state equipped with exactly two v-coloured arrows. Each arrow is placed on one of the positions indicated below and may point in either direction.



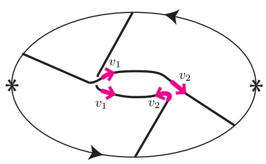
A *graph state* s of F is a choice of vertex state at each vertex of F, and an *arrow marked graph state* $\vec{s}$ of F is a choice of arrow marked vertex state at each vertex of F. Note that each graph state corresponds to a specific family of disjoint cycles in F, and this family is independent of embedding (although different embeddings of F will generally use different vertex states to generate the same family of disjoint cycles).

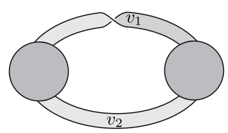
Definition 4.3. Let F be a 4-regular embedded graph, and let $\vec{s}$ be an arrow marked graph state of F. Regard $\vec{s}$ as an arrow presentation of an embedded graph by viewing each component of $\vec{s}$ as a circle marked by the labelled arrows arising from the arrow marked vertex states. Denote this new graph by $F_{\vec{s}}$, and because the vertices of $F_{\vec{s}}$ arise from a family of disjoint cycles of F, we call $F_{\vec{s}}$ a *cycle family graph* of F. Also note that there is a natural identification between the vertex set of F the edge set of $F_{\vec{s}}$. We denote the set of all cycle family graphs of a 4-regular embedded graph F by $\mathcal{C}(F)$.

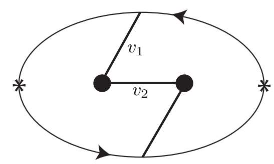
Figure 7. Equivalent arrowed vertex states, with flat arrows on the left, and twisted arrows on the right.

Example 4.4. As an example, if $F =$ 

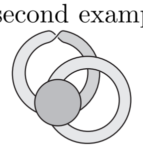
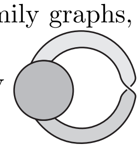
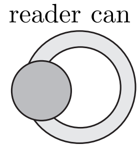
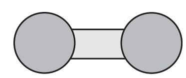
, then one of the arrow marked graph state $\vec{s}$ of F gives the following cycle family graph:

$F_{\vec{s}} =$ 

$=$ 

$=$ 

.

Example 4.5. As a second example of cycle family graphs, the reader can verify that the complete set of cycle family graphs of  contains only  ,  and  .

We will say that two arrow marked vertex states are *equivalent* if we can obtain one from the other by reversing the direction of pairs of arrows with the label. Figure 7 illustrates this in the case of splits. We refer to the arrowed states on the left of Figure 7, with the arrows pointing in opposite directions, as *flat* arrowed states, and the arrowed states on the right of Figure 7, with the arrows pointing in the same direction, as *twisted* arrowed states. Furthermore, we say that two arrow marked graph states $\vec{s}$ and $\vec{s}'$ of F are *equivalent* if for each vertex of F, the arrow marked vertex states of $\vec{s}$ and $\vec{s}'$ at that vertex are equivalent when thought of in terms of arrow presentations.

Lemma 4.6. *If F is a 4-regular embedded graph and $\vec{s}$ and $\vec{s}'$ are two equivalent arrow marked graph states of F, then $F_{\vec{s}} = F_{\vec{s}'}$ as embedded graphs.*

Proof. We need to show that the arising arrow presentations, $F_{\vec{s}}$ and $F_{\vec{s}'}$ are equivalent, that is, $F_{\vec{s}'}$ can be obtained from $F_{\vec{s}}$ by reversing the direction of some of the pairs of arrows of the same colour and by homeomorphism of the cycles. To do this, it is enough to show that the arrow marked graph state $\vec{s}'$ can be obtained from $\vec{s}$ by reversing the direction of some of the pairs of arrows of the same colour and by homeomorphism of the cycles. The fact that this is indeed the case is easily verified by checking the defining relations of equivalent arrow marked graph and vertex states. □

We note that the converse of the above lemma is false as non-equivalent arrow marked graph states may result in equivalent cycle family graphs.

Corollary 4.7. *There are at most $6^{v(F)}$ distinct cycle family graphs of a 4-regular embedded graph F.*

At certain points in this paper we will be particularly interested *duality states*. These are arrowed states which arise by restricting all of the vertex states to splits with flat arrows. In particular, Tait graphs arise from special duality states, as follows.

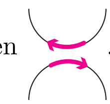
Proposition 4.8. *Let F be a checkerboard coloured embedded 4-regular graph. Then there exist arrow marked graph states $\vec{b}$ and $\vec{w}$ of F such that $F_{\vec{b}} = F_{bl}$ and $F_{\vec{w}} = F_{wh}$. Moreover, both $\vec{b}$ and $\vec{w}$ are duality states.*

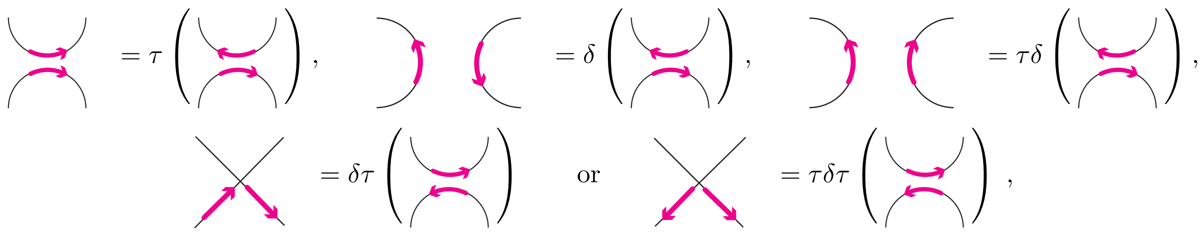
Proof. Construct an arrow marked graph state of F by choosing the duality state consisting of flat splits where the split at each vertex results in arcs that follow the boundaries of the black regions at that vertex. The resulting ribbon graph $F_{\vec{b}}$ is exactly the Tait graph F_{bl} since the cycles follow the boundaries of the

black regions (giving one vertex in each black region), and, since the arrow marked state is flat, there is an edge added whenever two black regions meet at a vertex as prescribed by the definition of F_{bl}. The whiteface result is proved analogously by choosing the flat splits at each vertex that follow the boundaries of the white regions. $\square$

4.2. **Twisted duals and cycle family graphs.** We are now ready for the first of our main theorems. We will begin by showing that all of the cycle family graphs of a 4-regular embedded graph are twisted duals, thus generalizing the well known property of Equation 4.1. We will then prove the converse of this result, that twisted duals are exactly the cycle family graphs of a medial graph. This converse generalizes Proposition 4.1.

Theorem 4.9. *If F is a 4-regular embedded graph and $F_{\vec{s}}$ and $F_{\vec{s}'}$ are two cycle family embedded graphs, then $F_{\vec{s}}$ and $F_{\vec{s}'}$ are twisted duals.*

Proof. It is enough to show that if the arrow marked graph states $\vec{s}$ and $\vec{s}'$ differ at exactly one vertex then $F_{\vec{s}}$ and $F_{\vec{s}'}$ are twisted duals. To show this, assume that $\vec{s}$ and $\vec{s}'$ differ at the vertex $v \in V(F)$ and that e_v is the label of the edge in $F_{\vec{s}}$ and the corresponding edge in $F_{\vec{s}'}$ that arises from the pair of v-coloured arrows in $\vec{s}$ and $\vec{s}'$. We then need to show that $\xi(F_{\vec{s}}, e_v) = (F_{\vec{s}'}, e_v)$ for some $\xi \in \mathfrak{G}$. To show this, we may assume with out loss of generality (as $\mathfrak{G}$ is a group) that the arrow marked vertex state at v in $\vec{s}$ is a split with flat arrow markings. Locally, (the arrow presentation of) $F_{\vec{s}}$ is then . Then, in $\vec{s}'$ the arrow presentation of $F_{\vec{s}'}$ is locally one of



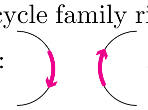
where the arrow presentations are all identical outside of the region shown in the diagrams. Thus we have shown that when $\vec{s}$ and $\vec{s}'$ differ at v, then $\xi(F_{\vec{s}}, e_v) = (F_{\vec{s}'}, e_v)$ for some $\xi \in \mathfrak{G}$, as required. $\square$

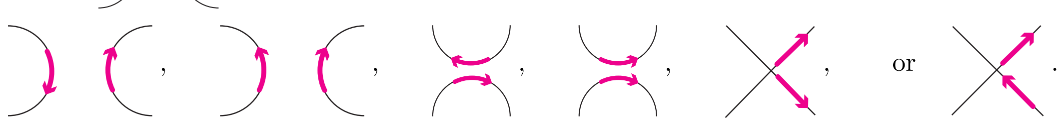
As mentioned above, Theorem 4.10 will give the converse of Theorem 4.9: all partial duals are cycle family graphs of a 4-regular embedded graph. Theorems 4.9 and 4.10 thus together give a characterization of the orbits of the ribbon group action: $Orb(G) = \mathcal{C}(G_m)$. This means that the ribbon group action on an embedded graph G generates precisely the set of cycle family graphs of medial graph of G.

Theorem 4.10. *If G is an embedded graph, then the set of cycle family graphs of G_m is precisely the set of all twisted duals of G,* i.e.

$$\mathcal{C}(G_m) = Orb(G).$$

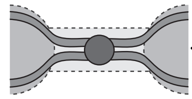
Proof. By Theorem 4.9, if $H \in \mathcal{C}(G_m)$, then $H \in Orb((G_m)_{\vec{s}}))$ for any arrow marked state $\vec{s}$. In particular this is true for the arrow marked state $\vec{b}$ corresponding to the black face graph that is guaranteed by Proposition 4.8, so that $(G_m)_{\vec{b}} = (G_m)_{bl} = G$. Thus $\mathcal{C}(G_m) \subseteq Orb(G)$.

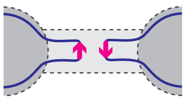
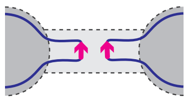
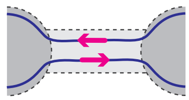
To show that $\mathcal{C}(G_m) \supseteq Orb(G)$, we need to show that if G and G' are twisted duals of one another, then they are both cycle family ribbon graphs of the medial ribbon graph G_m of G. To show this, consider locally an edge e of G: . At the edge e, the twisted dual will take on one of the following six forms.

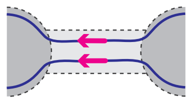
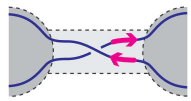
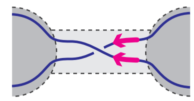


To prove the result it is enough to show that each of these six forms arise as an arrow marked vertex state of $v_e \in V(G_m)$. (This is since the local configurations at the edges of the arrow presentation of G are connected

to each other in the same way that the local configurations at the vertices of G_m are connected to each other to form the cycles.)

Now the vertex v_e of G_m is embedded in the edge e of G thus: . We then see that the possible arrow marked vertex states of G_m at v_e are

, , ,

, , and ,

as required. Thus, if $H \in Orb(G)$, then H is a twisted dual of G, and by the above $H \in \mathcal{C}(G_m)$ and $Orb(G) \subseteq \mathcal{C}(G_m)$. Thus $Orb(G) = \mathcal{C}(G_m)$ as desired. □

We note that Theorem 4.9 does not follow from Theorem 4.10 since, unlike the planar case, there are 4-regular embedded graphs that do not arise as the medial graph of an embedded graph. (To see this, observe that there are three embedded graphs with exactly one edge, giving rise to three medial graphs with two edges. However, as there are more than three 4-regular embedded graphs with two edges, not every 4-regular embedded graph can be an embedded medial graph.)

It follows immediately from Proposition 4.8 that Theorem 4.10 generalizes Equation 4.1 and Propositions 4.1, since $(G_m)_{bl} = (G_m)_{\vec{b}}$ and $(G_m)_{wh} = (G_m)_{\vec{w}}$ are both in $\mathcal{C}(G_m)$, and G and $G^* = G^{\delta(E(G))}$ are both in $Orb(G)$.

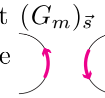
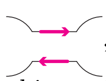
We have seen the relation between cycle family graphs and twisted duals. Since partial duality is a special case of twisted duality that is of independent interest, we now specialize our results to partial duality.

Theorem 4.11. *(i) If F is a 4-regular ribbon graph and $F_{\vec{s}}$ and $F_{\vec{s}'}$ are cycle family ribbon graphs, for some $\vec{s}$ and $\vec{s}'$ that are duality states, then $F_{\vec{s}}$ and $F_{\vec{s}'}$ are partial duals.*

(ii) If G and G' are partial duals, and G_m is the embedded medial graph of G, then there are duality states $\vec{s}$ and $\vec{s}'$ such that $G = (G_m)_{\vec{s}}$ and $G' = (G_m)_{\vec{s}'}$.

(iii) $(G_m)_{\vec{s}}$ is a partial dual of G if and only if $\vec{s}$ is a duality state.

Proof. The first two parts of the theorem can be proved by following the proof of Theorem 4.10 and restricting to partial duality and duality states. Due to this similarity the proofs are omitted.

The proof of (iii) is as follows. Since G_m is a 4-regular ribbon graph and $G = (G_m)_{\vec{s}}$ for some duality state $\vec{s}$, necessity follows by (i). For sufficiency, we may assume without loss of generality that $(G_m)_{\vec{s}}$ is obtained from G by forming the partial dual at a single edge e (so $(G_m)_s = G^{\delta(e)}$). If e is the edge , then the corresponding edge in $(G_m)_{\vec{s}}$ is , where the graphs are identical except in the region shown. It is easily seen that the only way that this configuration can arise from an arrow marked vertex state at $v_e \in V(G_m)$ is if the arrowed vertex state is a flat split. □

4.3. **Medial graphs and cycle family graphs.**

In this subsection we will prove another of our main theorems, this one generalizing Equation 4.2 and Proposition 4.2. The theorem shows that cycle family graphs extend the essential relations among plane graphs and their medial and Tait graphs. Furthermore, we show that if F is a 4-regular graph, then its set of cycle family graphs is precisely the set of embedded graphs that have embedded medial graphs isomorphic to F as abstract graphs. We actually prove a stronger result: we not only show that the set of cycle family graphs of F give all the embedded graphs whose medial graphs are isomorphic to F as abstract graphs, but we also give specific conditions for when a medial graph is simply a twist of F, that is, a different locally admissible embedding of the same combinatorial map (these were defined in Subsection 3.3).

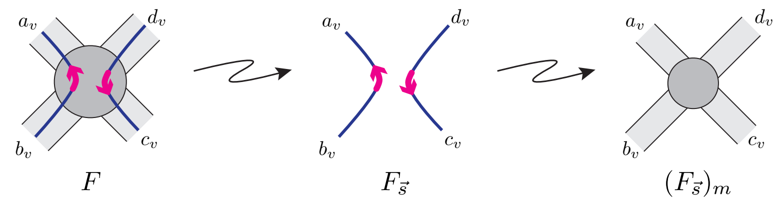
Theorem 4.12. *If F is a 4-regular embedded graph and $\vec{s}$ is an arrow marked graph state of F, then:*

(i) if $\vec{s}$ is a duality state, $(F_{\vec{s}})_m$ and F are twists of one another, i.e. $(F_{\vec{s}})_m = F^{\tau(A)}$ *for some $A \subseteq E(F)$, and hence $(F_{\vec{s}})_m$ and F are isomorphic as abstract graphs;*

(ii) otherwise $(F_{\vec{s}})_m$ and F are isomorphic as abstract graphs, but not necessarily twists of one another.

Proof. In order to prove the statements it is enough to consider what happens locally at a vertex v of F in the formation of $(F_{\vec{s}})_m$. We assume that v is incident with edges labelled a_v, b_v, c_v, d_v in the cyclic order $(a_v\, b_v\, c_v\, d_v)$

To prove the first statement, assume that $\vec{s}$ is a duality state. Without loss of generality we may assume that the cycles defining the cycle family graph $F_{\vec{s}}$ travel between edges a_v and b_v and between c_v and d_v. This is shown as the first step in the figure below.

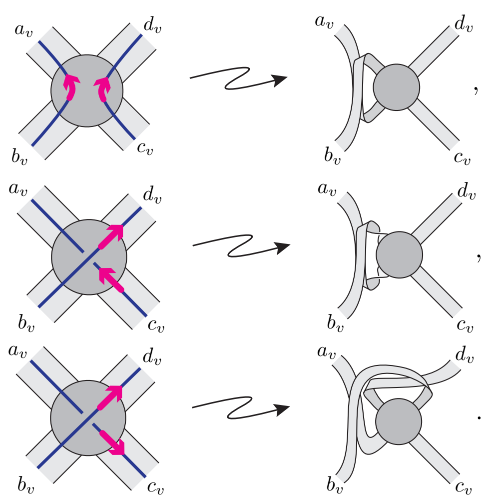


Now taking the medial graph of $F_{\vec{s}}$ will add a vertex incident with edges labelled a_v, b_v, c_v, d_v with the cyclic order $(a_v\, b_v\, c_v\, d_v)$ as shown in the figure above possibly with additional twisting of the edges. Note that since we do not know if the arcs (a_v, b_v) and (c_v, d_v) are connected to each other, we do not know if the edge e_v of the cycle family graph will embed in the vertex v shown the first figure. This means that when we form the medial graph $(F_s)_m$, as shown in the third step of the figure, we know nothing about the twisting of the edges. The figure is slightly misleading in this respect.

Finally, since up to twisting of the edges, F and $(F_{\vec{s}})_m$ are identical at the vertex $v \in F$ and the corresponding vertex in $(F_{\vec{s}})_m$ for each vertex v of F and the endpoints marked a_v, b_v, c_v, d_v in the figure are connected in the same way in F and in $(F_{\vec{s}})_m$, it follows that these two embedded graphs, when viewed as arrow presentations, have the same sequence of labels on the vertex disc, although possibly not the same directions of the arrows. Thus, $(F_{\vec{s}})_m = F^{\tau(A)}$ for some $A \subseteq E(G)$.

This completes the proof of the first statement.

The second statement is proven similarly: all that is needed to adapt the proof is to check that the remaining vertex states at v lead to an appropriate configuration at the corresponding vertex in $(F_{\vec{s}})_m$. This is seen by the following calculations:



There is a natural identification between the vertex set of F and the edge set of $F_{\vec{s}}$, and a natural identification between the edges set of $F_{\vec{s}}$ and the vertex set of $(F_{\vec{s}})_m$, and hence a natural identification between the vertices of F and the vertices of $(F_{\vec{s}})_m$. Thus, the second statement follows from observing that these local configurations mean that a pair of vertices are connected by an edge in F if and only if they are connected by an edges in $(F_{\vec{s}})_m$. □

Again by Proposition 4.8, it is clear that Theorem 4.12 generalizes Equation 4.2 and Proposition 4.2, since $F_{bl} = F_{\vec{b}}$ and $F_{wh} = F_{\vec{w}}$ for the duality states $\vec{b}$ and $\vec{w}$.

Proposition 4.13. *Let G and H be 4-regular embedded graphs. If G and H are isomorphic as abstract graphs, then $\mathcal{C}(G) = \mathcal{C}(H)$. In particular, all embeddings of a given 4-regular graph generate the same set of cycle family graphs.*

Proof. Suppose the isomorphism of the underlying abstract graphs of G and H is via $f : E(G) \to E(H)$. Then an arrow marked state $\vec{s}$ of G induces an arrow marked state $f(\vec{s})$ of H by pairing edges $f(a)$ and $f(b)$ with an arrow from $f(a)$ to $f(b)$ in $f(\vec{s})$ if and only if $\vec{s}$ pairs edges a and b with an arrow from a to b. Thus the arrow marked disks of $G_{\vec{s}}$ are precisely the arrow marked disks of $H_{f(\vec{s})}$ under the relabelling given by f. Hence $G_{\vec{s}} = H_{f(\vec{s})} \in \mathcal{C}(H)$, and $\mathcal{C}(G) \subseteq \mathcal{C}(H)$. A symmetric argument shows that $\mathcal{C}(H) \subseteq \mathcal{C}(G)$. □

With the following theorem we are now able to answer the original question of finding the exact set of embedded graphs that have medial graphs isomorphic to a given 4-regular graph F.

Theorem 4.14. *If F is any 4-regular graph, then the set of cycle family graphs of any embedding of F is the precisely the set of all embedded graphs G such that G_m and F are equivalent as abstract graphs,* i.e. *if $\widetilde{F}$ is any embedding of F, then*

$$\mathcal{C}(\widetilde{F}) = \{G | G_m \cong \widetilde{F}\} = \{G | G_m \cong F\}.$$

Proof. If $G \in \mathcal{C}(F)$, then $G = \widetilde{F}_{\vec{s}}$ for some state $\vec{s}$ and hence $G_m = (\widetilde{F}_{\vec{s}})_m \cong \widetilde{F} \cong F$, by Theorem 4.12.

Conversely, if $G_m \cong \widetilde{F} \cong F$, then G_m is equivalent to F' as embedded graphs, for some embedding F' of F. By Proposition 4.8, $G = (G_m)_{\vec{b}} = F'_{\vec{b}} \in \mathcal{C}(F')$. Finally, by Proposition 4.13, we have $\mathcal{C}(F') = \mathcal{C}(\widetilde{F})$, completing the proof.

□

The above theorem can be used to characterize 4-regular embedded graphs that are isomorphic as graphs in terms of cycle family graphs:

Corollary 4.15. *Let F and F' be two 4-regular embedded graphs, then F and F' are isomorphic as abstract graphs (that is $F \cong F'$) if and only if they admit the same set of cycle family graphs $\mathcal{C}(F) = \mathcal{C}(F')$.*

Proof. The result follows since, by Theorem 4.14

$$\mathcal{C}(F) = \{G | G_m \cong F\} = \{G | G_m \cong F'\} = \mathcal{C}(F').$$

□

A notable special case corollary 4.15 occurs when F and F' are checkerboard colourable.

Corollary 4.16. *Let F and F' be 4-regular, checkerboard colourable, embedded graphs, with $F \cong F'$, then the four Tait graphs, F_{wh}, F_{bl}, F'_{wh} and F'_{bl}, of F and F', are all twisted duals of one another.*

Proof. This follows immediately from Corollary 4.15 and Theorem 4.9. □

Theorem 4.14 also provides a second characterization of the orbit of an embedded graph under the ribbon group action:

Corollary 4.17. *If G is an embedded graph and $\vec{s}$ is an arrow marked state of the embedded medial graph G_m, then $((G_m)_{\vec{s}})_m \cong G_m$,* i.e.

$$Orb(G) = \{H : H_m \cong G_m\}.$$

Proof. This follows immediately from Theorems 4.10 and 4.14.

□

To illustrate Corollary 4.17, let G denote the plane digon. The orbit of G is shown in Example 3.8. It is readily verified that every embedded graph in $H \in Orb(G)$ has a medial graph isomorphic to G_m. On the other hand, if H has a medial graph isomorphic to G_m (so $H \in \{H : H_m \cong G_m\}$), then H must have two edges. By calculating the medial graphs of each embedded graph with two edges, one can easily check that if $H \in \{H : H_m \cong G_m\}$ then $H \in Orb(G)$

Remark 4.18. Returning for a moment to the motivating application in self-assembling DNA nanostructures, we note that if a 4-regular graph F can be assembled out of DNA strands, with relatively small faces, then it might be used as a template to construct all of the graphs in $\mathcal{C}(F)$. The branched junction molecules forming its vertices would be decomposed into various vertex states, giving a state $\vec{s}$ of F. This kind of split at the vertices was done to find Hamilton circuits by Adelman [1]. Then long (relative to the faces of F) double strands of DNA with 'sticky ends' (*i.e.* an extended single strand of unsatisfied bases) might be introduced to then form the edges of $F_{\vec{s}}$, with the relatively small cycles of $\vec{s}$ as the vertices.

4.4. **Medial graphs and partial duals.** As mentioned previously, Euler-Poincaré duality can be completely characterized in terms of equivalence of embedded medial graphs:

$$\{G, G^*\} = \{H \mid H_m = G_m\}.$$

We have seen above that the set of twisted duals of an embedded graph G arises as the set of graphs with the same medial graph as G when the medial graphs are considered as abstract graphs:

$$\{G^{\xi(A)} \mid \xi \in \mathfrak{G},\ A \subseteq E(G)\} = Orb(G) = \{H \mid H_m \cong G_m\}.$$

From these result, we can posit that notions of duality are generated by notions of graph equivalence, that is we can take the point of view that the set $\{H \mid H_m \sim G_m\}$ describes a set of generalized dual graphs for each graph equivalence $\sim$. In this section we consider the the duality generated by considering H_m and G_m to be equivalent as combinatorial maps. In particular we will show that

$$\{G^{\delta(A)} \mid A \subseteq E(G)\} = \{G | G_m \text{ and } F \text{ are equivalent as combinatorial maps }\}$$

and therefore equivalence as combinatorial maps induces the concept of partial duality.

Let F be a 4-regular embedded graph. We will denote the subset of cycle family graphs of F that are obtained from duality states by $\mathcal{C}_\delta(F)$, that is

$$\mathcal{C}_\delta(F) := \{F_{\vec{s}} \mid \vec{s} \text{ is a duality state of } F\}.$$

The following theorem states that the set $\mathcal{C}_\delta(F)$ of cycle family graphs generated by duality states completely characterizes the set of embedded graphs with a medial graph that are equivalent to F as combinatorial maps. The theorem should be compared with Theorem 4.14.]

Theorem 4.19. *If F is any 4-regular embedded graph, then the subset of cycle family graphs of F that generated by all duality states by is precisely the set of all embedded graphs G such that G_m and F are equivalent as combinatorial maps,* i.e.

$$\begin{aligned}\mathcal{C}_\partial(F) &= \{G | G_m \textit{ and } F \textit{ are equivalent as combinatorial maps }\}\\ &= \{G \mid (G_m)^{\tau(A)} = F, \textit{ for some } A \in E(G_m)\}.\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 4.14. If $G \in \mathcal{C}_\delta(F)$, then $(G_m)^{\tau(A)} = F$, for some $A \in E(G_m)$, by the second part of Theorem 4.12. On the other hand, if G is an embedded graph such that G_m and F are equivalent as combinatorial maps, then G_m is checkerboard colourable with the black faces containing the vertices of G. We take the arrowed state $\vec{b}$ of G_m guaranteed by Proposition 4.8 so that $(G_m)_{\vec{b}} = (G_m)_{bl} = G$. Since $(G_m)^{\tau(A)} = F$, for some $A \in E(G_m)$, by twisting some of the edges of G_m, the arrowed state $\vec{b}$ of G_m induces an arrow marked state $\vec{\beta}$ in F. This induced arrow marked state is a duality state since $\vec{b}$ was. Moreover, $G = F_{\vec{\beta}}$ since $\vec{b}$ and $\vec{\beta}$ are equivalent as arrow presentations. Thus $G \in \mathcal{C}_\delta(F)$ as required. □

Theorem 4.19 can be used to characterize 4-regular embedded graphs that are equivalent as combinatorial maps:

Corollary 4.20. *Let F and F' be two 4-regular embedded graphs, then F and F' are equivalent as combinatorial maps if and only if $\mathcal{C}_\delta(F) = \mathcal{C}_\delta(F')$.*

Proof. The result follows since, by Theorem 4.19

$$\mathcal{C}_\delta(F) = \{G | (G_m)^{\tau(A)} = F, \text{ for some } A \in E(G_m)\} = \{G | (G_m)^{\tau(A)} = F', \text{ for some } A \in E(G_m)\} = \mathcal{C}_\delta(F').$$

□

In parallel with Corollaries 4.15 and 4.16, a special case of Corollary 4.20 occurs when F and F' are checkerboard colourable.

Corollary 4.21. *Let F and F' be 4-regular, checkerboard colourable, embedded graphs, with F and F' are equivalent as combinatorial maps, then the four Tait graphs, F_{wh}, F_{bl}, F'_{wh} and F'_{bl}, of F and F', are all partial duals.*

Proof. This follows immediately from Corollary 4.20 and Theorem 4.11. □

Theorem 4.19 provides a characterization of the orbit of an embedded graph under the action of the subgroup of the ribbon group generated by δ. We will denote this orbit by $Orb_{\langle\delta\rangle}(G)$. The orbit is equal to the set of partial duals of G.

Corollary 4.22. *If G is an embedded graph and $\vec{s}$ is an duality state of the embedded medial graph G_m, then $((G_m)_{\vec{s}})_m$ and G_m are equivalent as combinatorial maps,* i.e.

$$\begin{aligned} Orb_{\langle\delta\rangle}(G) &= \{H \mid G_m \text{ and } H_m \text{ are equivalent as combinatorial maps}\} \\ &= \{G \mid (G_m)^{\tau(A)} = H_m, \text{ for some } A \in E(G_m)\}. \end{aligned}$$

Proof. This follows immediately from Theorems 4.10 and 4.14.

□

Using the relationships among partial duals, twisted duals and medial graphs, we can apply results of Las Vergnas ([59]) and Kotzig ([53]) to deduce some properties of the orbits under the ribbon group action.

Corollary 4.23. *Let G be a plane graph. Then*

$$\max\{v(H) \mid H \in Orb(G)\} = \max\{v(H) \mid H \in Orb_{\langle\delta\rangle}(G)\}.$$

Proof. Las Vergnas' Proposition 6.1 from [59], implies that the maximum number of circuits in any duality state of G_m is equal to the maximum number of circuits in any state of G_m. Since the cycles in the states of G_m form the vertices of the cycle family graphs we have

$$\max\{v(H) \mid H \in \mathcal{C}_{\langle\delta\rangle}(G_m)\} = \max\{v(H) \mid H \in \mathcal{C}(G_m)\}.$$

The result then follows since $\mathcal{C}_{\langle\delta\rangle}(G_m) = Orb_{\langle\delta\rangle}(G)$, by Theorem 4.19 and Corollary 4.22; and since $\mathcal{C}(G_m) = Orb(G)$, by Theorem 4.14 and Corollary 4.17. □

The following corollary relates the number of spanning trees of a ribbon graph G and of its dual G^* to the number of blossoms (*i.e.* one vertex embedded graphs) in $Orb_{\langle\delta\rangle}(G)$.

Corollary 4.24. *Let G be an graph embedded in the plane, the torus or the real projective plane. Then the number of blossoms in $Orb_{\langle\delta\rangle}(G)$ is bounded above by the total number of spanning trees in G and G^*.*

Proof. By Las Vergnas' Corollary 2.4 from [59], every duality state of G_m that contains exactly one cycle corresponds to a spanning in G or G^*. (The plane case of this result is due to Kotzig [53].) Every duality state gives rise to a (not necessarily distinct) cycle family graph in $\mathcal{C}_\delta(G_m)$. Thus

$$|\{\text{spanning trees of } G \text{ or } G^*\}| \geq |\mathcal{C}_\delta(G_m)| = |Orb_{\langle\delta\rangle}(G)|,$$

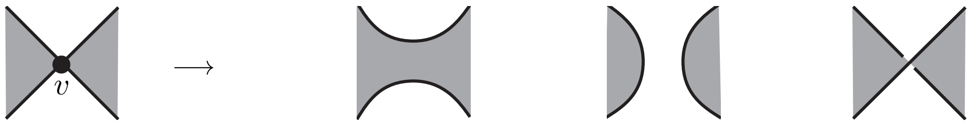
where the last equality follows from Theorem 4.19 and Corollary 4.22. □

The above result can also be extended to quasi-trees. From [24] (see also [14, 15, 86]), a *quasi-tree* is an embedded graph with exactly one boundary component (or face).

Corollary 4.25. *Let G be an embedded graph, then the number of blossoms in $Orb_{\langle\delta\rangle}(G)$ is bounded above by the number of spanning quasi-trees of G.*

Proof. From [68], if $A \subseteq E(G)$, and $A^c = E(G)\backslash A$, then the number of vertices of $G^{\delta(A)}$ is equal to the number of boundary components of $G\backslash A^c$. It then follows that $G^{\delta(A)}$ is a blossom if and only if $G\backslash A^c$ is a spanning quasi-tree. The result then follows, noting that the partial duals need not be distinct. □

4.5. **Cycle family graphs and checkerboard colourings.** Here we define an action of the ribbon group $\mathfrak{G}^n$ on an arrow marked vertex state of an embedded medial graph G_m. Obviously, G_m could equivalently be any 4-regular checkerboard colourable embedded graph, but we will typically apply these results to medial graphs. In order to do this we need to be able to distinguish among the three vertex states. If we canonically checkerboard colour G_m, then we can distinguish among the vertex states at v as in the following figure. Here, following knot theory conventions, the graphs are identical outside these local neighbourhoods.



in G_m white split black split crossing.

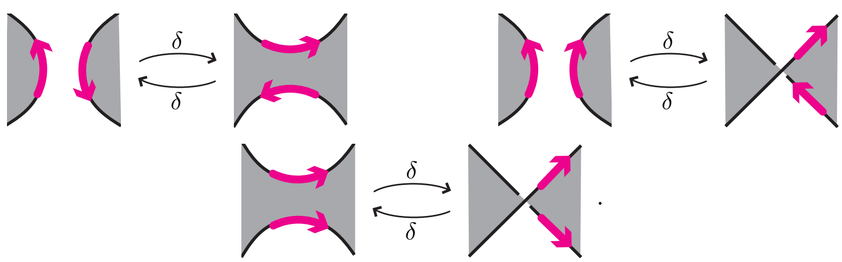
We denote the graphs that result from each of these vertex states as follows:

$(G_m)_{wh(v)}$ is the embedded graph that results from taking the white split state at v,
$(G_m)_{bl(v)}$ is the embedded graph that results from taking the black split state at v, and
$(G_m)_{cr(v)}$ is the embedded graph that results from taking the crossing state at v.

This kind of operation at a vertex appears in the literature under several names, including 'smoothing' and 'transition'. We use the terminology 'split' here to emphasize that the operation 'splits', or separates, the regions about a vertex.

We now define an action of the group $\mathfrak{G}^n$ on an arrow marked vertex state. Let $(G, \ell) \in \mathcal{G}_{or(n)}$. Then the order ℓ of the edges of G induces an order $\hat{\ell} = (v_1, \dots, v_n)$ of the vertex set of the embedded medial graph G_m. We let $(G_m, \hat{\ell})$ denote this canonically checkerboard coloured medial graph equipped with the induced vertex order from (G, ℓ). To define an action on the set of arrow marked states of G_m we let $\vec{s}$ be an arrow marked graph state of G_m and $(\vec{s}, i)$ be the vertex state at the i^{th} vertex v_i in the order $\hat{\ell}$. Then define $\tau(\vec{s}, i)$ to be the pair $(\vec{s}\,', i)$, where $\vec{s}\,'$ is obtained from $\vec{s}$ by reversing exactly one of the arrows of the arrow marked vertex state at v_i in $\vec{s}$.

Let $\delta(\vec{s}, i)$ be the pair $(\vec{s}\,', i)$ where $\vec{s}\,'$ is constructed by changing the arrow marked vertex state at v_i in $\vec{s}$ as specified by the following figure:



Next we want to extend this action to the set of states of G_m. Let $\mathcal{C}_{or}(G_m, \hat{\ell})$ denote the set of arrow marked graph states of $(G_m, \hat{\ell})$ with the vertex ordering induced by ℓ. Define an action of $\mathfrak{G}^n$ on $\mathcal{C}_{or}(G_m, \hat{\ell})$ by

$$\boldsymbol{\xi}(\vec{s}, \hat{\ell}) := (\xi_1, \xi_2, \xi_3, \dots, \xi_n)(\vec{s}, \hat{\ell}) = \xi_n(\cdots \xi_3(\xi_2(\xi_1(\vec{s}, v_1), v_2), v_3) \cdots), v_n),$$

where $(\xi_1, \xi_2, \xi_3, \dots, \xi_n) \in \mathfrak{G}^n$ and $\vec{s} \in \mathcal{C}_{or}(G_m, \hat{\ell})$.

Proposition 4.26. *The action of $\mathfrak{G}^n$ on $\mathcal{C}_{or}(G_m, \ell)$ described above is a group action. Moreover,*

$$(G_m, \hat{\ell})_{\boldsymbol{\xi}(\vec{s})} = \boldsymbol{\xi}((G_m, \hat{\ell})_{\vec{s}}),$$

where $\boldsymbol{\xi} \in \mathfrak{G}^n$.

Proof. To prove the first part of the proposition we need to check that $\delta^2(\vec{s}, v) = \tau^2(\vec{s}, v) = (\tau\delta)^3(\vec{s}, v) = 1$ and this is easily verified. The second part of the proposition is tautological when the arrowed states are viewed as arrow presentations of embedded graphs. $\square$

Just as the action of $\mathfrak{G}^n$ may be interpreted geometrically, Proposition 4.26 may be given from a geometric perspective as follows.

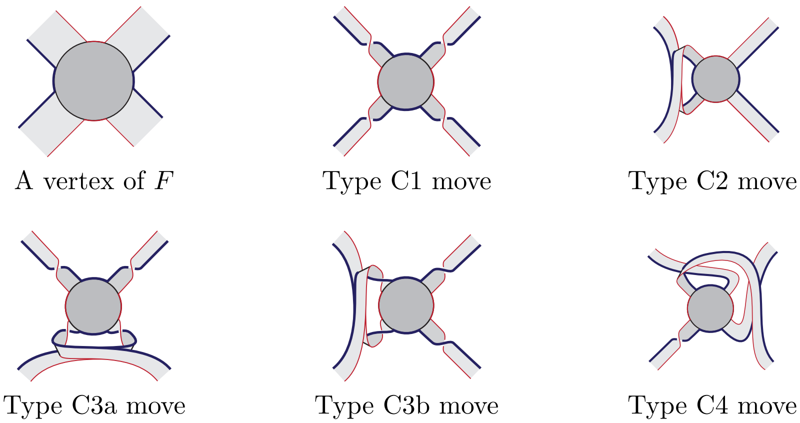


FIGURE 8. Moves on a checkerboard coloured embedded graph.

Proposition 4.27. *Let G be an embedded graph with embedded medial graph G_m, and index the vertices of G_m by $E(G)$. Let $\Gamma = \prod_{i=1}^{6} \xi_i(A_i)$ where the A_i's partition $E(G)$, and the ξ_i's are the six elements of $\mathfrak{G}$. If $\vec{s}$ is an arrow marked state of G_m, then let $\vec{s}^{\Gamma}$ be the result of applying $\xi_i(\vec{s}, v_e)$ if $e \in A_i$. Then*

$$(G_m)_{\vec{s}^{\Gamma}} = ((G_m)_{\vec{s}})^{\Gamma}.$$

4.6. **Orbits under the action induced by the subgroups of $\mathfrak{G}$.** We have given in Corollary 4.14 a characterization of the orbit of an embedded graph G under the entire ribbon group action in terms of medial graphs. We have also given in Corollary 4.22 a characterization of the orbit of G under the action subgroup generated by all elements of the form $(1, \dots, 1, \delta, 1, \dots, 1)$, again in terms of medial graphs. Here we will provide a geometric characterizations of the orbits under the action of some other important subgraphs of the ribbon group. These characterizations are implicit in Proposition 4.27. A full study of actions of the many subgroups of a ribbon group on n edges for some class of graphs opens a new and interesting area investigation.

The group $\mathfrak{G} =< \delta, \tau \mid \delta^2, \tau^t, (\delta\tau)^3 >$ has five non-trivial subgroups, the four proper subgroups being cyclically generated. As in Subsection 3.3, each of these subgroups define an action on the sets $\mathcal{G}_{or(n)}$. If $\xi \in \mathfrak{G}$ and $\langle \xi \rangle$ is the subgroup of $\mathfrak{G}$ generated by ξ, then we denote the orbits under this action of $\langle \xi \rangle^{e(G)}$ on $\mathcal{G}_{or(n)}$ by $Orb_{\langle \xi \rangle}(G, \ell) := \{\boldsymbol{\xi}(G, \ell) \mid \boldsymbol{\xi} \in \langle \xi \rangle^{e(G)} \leq \mathfrak{G}^{e(G)}\}$ of the group action. Once again, with slight abuse of terminology, we define

$$Orb_{\langle \xi \rangle}(G) := \{H : (H, \ell) \in Orb_{\langle \xi \rangle}(G, \ell) \text{ for some edge order } \ell\}.$$

In this notation $Orb_{\langle \delta \rangle}(G)$ is the set of partial duals of G, and $Orb_{\langle \tau \rangle}(G)$ is the set of twists of G.

In this section, we characterize goemetrically the orbits $Orb_{\langle \xi \rangle}(G)$, generated by each subgroup of $\mathfrak{G}$ in terms of medial graphs. In order to do this we need to introduce some further notions of the equivalence checkerboard coloured medial graphs.

Let F be a checkerboard coloured 4-regular graph. Type $C1$, $C2a$, $C2b$, $C3$ and $C4$ moves are defined in Figure 8. In this figure, a Ci move, say, replaces a checkerboard coloured 4-valent vertex of F with the checkerboard coloured 4-valent vertex shown labelled Ci in the table, where Ci is $C1$, $C2a$, $C2b$, $C3$ or $C4$. (Note that, due to the non-trivial embeddings of the graphs, in the figure, the checkerboard colouring is indicated using bold (for the black face) and lighter (for the white face) lines for the boundary components.) We say that two checkerboard coloured 4-regular graphs F and H are $C1$-equivalent, written $G \sim_{C1} H$, if F and H are related by a finite sequence of $C1$ moves; $C2$-equivalent, written $G \sim_{C2} H$, if F and H are related by a finite sequence of $C2$ moves; $C3$-equivalent, written $G \sim_{C3} H$, if F and H are related by a finite sequence of $C3a$ and $C3b$ moves; and $C4$-equivalent, written $G \sim_{C4} H$, if F and H are related by a finite sequence of $C4$ moves. It is easy to verify that Ci-equivalence, for $i = 1, 2, 3, 4$, is an equivalence relation.

Theorem 4.28. *Let G be an embedded graph. Then*

(i) $Orb(G) = \{H : H_m \cong G_m\}$*;*

(ii) $Orb_{\langle\delta\rangle}(G) = \{H \mid G_m \sim_{C1} H_m$ *w.r.t canonical checkerboard colouring*$\}$
$= \{H \mid G_m$ *and* H_m *are equivalent as combinatorial maps*$\}$ *;*
(iii) $Orb_{\langle\tau\rangle}(G) = \{H \mid G_m \sim_{C2} H_m$ *w.r.t canonical checkerboard colouring* $\}$
(iv) $Orb_{\langle\delta\tau\rangle}(G) = \{H \mid G_m \sim_{C3} H_m$ *w.r.t canonical checkerboard colouring*$\}$
(v) $Orb_{\langle\delta\tau\delta\rangle}(G) = \{H \mid G_m \sim_{C4} H_m$ *w.r.t canonical checkerboard colouring*$\}$

Proof. The first result is Corollary 4.17 and the second is Corollary 4.22.

The remaining results follow easily from Proposition 4.27. To see this we begin by observing that, by Proposition 4.8, $G = (G_m)_{\vec{b}}$. Then if $\xi \in \mathfrak{G}$, and $e \in E(G)$ we have, by Proposition 4.27,

$$G^{\xi(e)} = \left((G_m)_{\vec{b}}\right)^{\xi(e)} = (G_m)_{(\vec{b})^{\xi(e)}}.$$

Finally, the C1 move is the geometric realization of $(G_m)_{(\vec{b})^{\xi(e)}}$ when $\xi = \delta$, the C2 move is the geometric realization of $(G_m)_{(\vec{b})^{\xi(e)}}$ when $\xi = \tau$, the C3a when $\xi = \tau\delta$, the C3b when $\xi = \delta\tau$, and the C4 when $\xi = \delta\tau\delta$. □

The following theorem summarizes how the main notation of equivalence of embedded graphs and the notions of duality studied here generate each other.

Theorem 4.29. *Let* G *and* H *be an embedded graphs. Then*

(i) G_m *and* H_m *are equivalent as embedded graphs if and only if* G *and* H *are natural duals;*
(ii) G_m *and* H_m *are equivalent as combinatorial maps if and only if* G *and* H *are partial duals;*
(iii) G_m *and* H_m *are equivalent as graphs if and only if* G *and* H *are twisted duals;*

In Subsection 4.4 we had various results concerning $Orb_{\langle\tau\rangle}(G)$. Here we highlight some further basic properties of $Orb_{\langle\tau\rangle}(G)$.

Proposition 4.30. *Let* G *be an embedded graph. Then*

(i) $|Orb_{\langle\tau\rangle}(G)|$ *is bounded above by the number of cycles in* G*.*
(ii) $G^{\tau(E(G))} = G$ *if and only is* G *is bipartite.*

Proof. We will work in the language of ribbon graphs. Both results follow from the observations that $\tau(e)$ changes the ribbon graph by adding a half-twist to the edge e, so up to the equivalence of ribbon graphs, $\tau(A)$ can only act by adding or removing half-twists to cycles, for each $A \subseteq E(G)$. The first result then follows by observing that $\tau(A)$ can only act by changing the orientability of a set of cycles of G.

For the second result, observe that $G^{\tau(E(G))} = G$ if and only if adding a half-twist to each edge results in an equivalent ribbon graph. This happens if and only if an even number of half-twists is added to each cycle. In turn this happens if and only if each cycle is of even length, or equivalently if and only if G is bipartite. □

4.7. **Deletion, contraction, and the medial graph.** We now discuss how the operations of deletion and contraction interact with forming medial graphs. Deletion and contraction of non-loop edges of an embedded graph are defined much as for abstract graphs. Let G be an embedded graph. In the language of ribbon graphs it is clear that deleting any edge of G or contracting a non-loop edge will result in a ribbon graph. When working in the language of cellularly embedded graphs, one has to ensure that deleting or contracting an edge results in a cellularly embedded graph. In particular, deleting a bridge changes not only the number of components of a graph, but the number of components of the surface it is embedded in. Thus deletion and contraction is often best done by converting to the language of ribbon graphs, carrying out the operation, then converting back to the language of cellularly embedded graphs. Deletion and contraction for arrow presentations can be defined similarly. Note that G/e and $G^{\delta(e)} - e$ are equivalent.

However, contracting loops requires some care. We follow Bollobás and Riordan's definition from Section 7 of [10]. Let G be an embedded graph regarded as a ribbon graph and suppose e is a loop of G, with v the vertex of G incident to e. Then the ribbon subgraph $(\{v\}, \{e\})$ of G has either one or two boundary components (depending on whether e is a twisted loop or not). Then the ribbon graph G/e is formed from G by attaching a vertex-disc to each of the boundary components of $v \cup e$ in G, then deleting e and v. From this definition, it is not hard to see that when e is a loop it is also true that the ribbon graphs G/e

and $G^{\delta(e)} - e$ are equivalent. Chmutov observed this in [17] and defined the *contraction of an edge* e of an embedded graph G by setting

$$G/e := G^{\delta(e)} - e.$$

We emphasize the fact that if e is a non-twisted loop whose ends are adjacent to each other on the vertex on which they lie, then contraction creates a vertex. For example if G is the orientable embedded graph consisting of one vertex and one edge e, then G/e consists of two vertices and no edges. We also recal that the medial graph of an isolated vertex is again an isolated vertex.

The following proposition may be readily observed by viewing G and G_m as ribbon graphs as in Figure 4, noting how the medial graph is transformed under the indicated operations.

Proposition 4.31. *Let G be an embedded graph with embedded medial graph G_m, and e be an edge of G. Then*

(i) $(G_m)_{bl(v_e)} = (G-e)_m$;
(ii) $(G_m)_{wh(v_e)} = (G/e)_m$;
(iii) $(G_m)_{cr(v_e)}$ *and* $(G^{\tau(e)}/e)_m$ *are twists of each other.*

5. The transition polynomial

The generalized transition polynomial, $q(G; W, t)$, of [28] is a multivariate graph polynomial that generalizes Jaeger's transition polynomial [45]. The transition polynomial assimilates the Penrose polynomial and Kauffman bracket, and agrees with the Tutte polynomial via a medial graph construction. We will now adapt $q(G; W, t)$ to embedded graphs and determine its interaction with the ribbon group action. This will allow us, in Section 6, to generalize the Penrose polynomial for plane graphs to embedded graphs and determine new properties for it. Also, in Section 7, by leveraging the relation determined in [33, 34] between the generalized transition polynomial and the topological Tutte polynomial of Bollobás and Riordan ([9, 10]), we determine new properties of the latter.

5.1. **The topological transition polynomial.**

The generalized transition polynomial, $q(G; W, t)$ extends the transition polynomial of Jaeger [44] to arbitrary Eulerian graphs and incorporates pair and vertex state weights. For the current application, however, we will restrict q to 4-regular embedded graphs (typically medial graphs) and we will only work in the generality needed for our current application. For example, since we will not use pair weights here, we give the weight systems simply in terms of vertex state weights. If an applications arises in the future where the pair weights are needed, because of the restriction to 4-regular graphs, they can be taken to be square roots of the vertex state weights. We refer the reader to [28] or [34] for further details.

A *weight system*, $W(F)$, of any 4-regular graph F (embedded or not) is an assignment of a weight in a unitary ring $\mathcal{R}$ to every vertex state of F. (We simply write W for $W(F)$ when the graph is clear from context.) If s is a state of F, then the *state weight* of s is $\omega(s) := \prod_{v \in V(F)} \omega(v, s)$, where $\omega(v, s)$ is the vertex state weight of the vertex state at v in the graphs state s. Note that a state s consists of a set of disjoint closed curves, and we refer to these as the components of the state, denoting the number of them by $c(s)$.

Definition 5.1. Let F be a 4-regular graph having weight system W with values in a unitary ring $\mathcal{R}$. Then the state model formulation of the *generalized transition polynomial* is

$$q(F; W, t) = \sum_{s} \omega(s) t^{c(s)},$$

where the sum is over all graph states s of F.

We now restrict our attention further to embedded medial graphs and particular weight systems determined by the embeddings. Because of these restrictions, we will call the generalized transition polynomial specialized for this application the *topological transition polynomial*, and define it as follows.

Definition 5.2. Let G be an embedded graph with embedded medial graph G_m. Define the *medial weight system* $W_m(G_m)$ using the canonical checkerboard colouring of G_m as follows. A vertex v has state weights given by an ordered triple $(\alpha_v,\ \beta_v,\ \gamma_v)$, indicating the weights of the white split, black split, and crossing

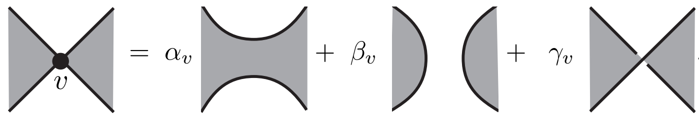
state, in that order. We write $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ for the set of these ordered triples, indexed equivalently either by the vertices of G_m or by the edges of G. Then the *topological transition polynomial* of G is:

$$Q(G,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) := q(G_m;W_m,t).$$

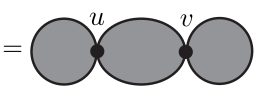
Proposition 5.3. *(see* [33]*) The topological transition polynomial may also be computed by repeatedly applying the following linear recursion relation at each $v \in V(G_m)$, and, when there are no more vertices of degree 4 to apply it to, evaluating each of the resulting closed curves to t:*

$$q(G_m,W_m,t) = \alpha_v q((G_m)_{wh(v)},W_m,t) + \beta_v q((G_m)_{bl(v)},W_m,t) + \gamma_v q((G_m)_{cr(v)},W_m,t).$$

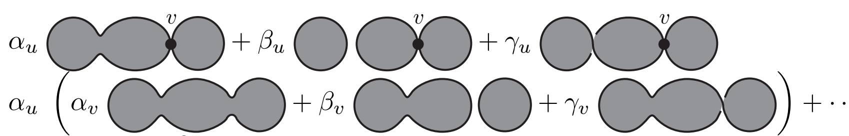
Pictorially, this is:



Example 5.4. For example, if $G =$ 

, then $G_m =$ 

and so

$Q(G_m;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) =$ 

$$= \alpha_u\alpha_v t + \alpha_u\beta_v t^2 + \alpha_u\gamma_v t + \cdots$$

Theorems 5.8 and 5.6 give properties of the topological transition polynomial that will allow us to easily manipulate related polynomials such as the topological Penrose polynomial of Section 6 and the topochromatic polynomial of Section 7. Theorem 5.6 shows precisely how the topological transition polynomial interacts with the ribbon group action, and Theorem 5.8 is a modified deletion/contraction reduction for well-behaved edges.

5.2. **Twisted duals and the transition polynomial.** The group $\mathfrak{G} = \langle \delta, \tau \mid \delta^2, \tau^2, (\tau\delta)^3 \rangle$ is isomorphic to S_3 via

$$\eta : \tau \mapsto (1\ 3) \quad \text{and} \quad \eta : \delta \mapsto (1\ 2).$$

Furthermore, the symmetric group S_3 acts on the ordered triple of the weight system at a vertex by permutation. This action by S_3 on a vertex can be extended to an action of S_3^n on medial graphs with n linearly ordered vertices. This can be done by mimicking the approaches used in Subsections 3.1 and 4.5. We will not formally define this action here and instead define the order independent analogue of the action, which is more convenient for our applications. This allows us to use the ribbon group to modify the medial weight system of an embedded medial graph.

Definition 5.5. Let G_m be a canonically coloured embedded medial graph of an embedded graph G with medial weight system W_m (or equivalently $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$), and vertices indexed by the edges of G. Let $\Gamma = \prod_{i=1}^{6} \xi_i(A_i)$ where the A_i's partition $E(G)$, and the ξ_i's are the six elements of $\mathfrak{G}$. Then ${W_m}^{\Gamma}$ (or $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})^{\Gamma}$), the *weight system permuted by* Γ, has the ordered triple of the weight system at a vertex v_e given by $\eta(\xi_i)(\alpha_{v_e},\beta_{v_e},\gamma_{v_e})$ when $e \in A_i$.

The classical Tutte polynomial has the duality property that the Tutte polynomial of a plane graph G is the same as that of its planar dual G^* with the roles of the variables x and y permuted. In [33, 34] it was shown that the topological transition polynomial has the duality property that

$$q(G_m;W_m,t) = q(G_m^*;W_m^*,t), \tag{5.1}$$

or equivalently,

$$Q(G;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) = Q(G^*;(\boldsymbol{\beta},\boldsymbol{\alpha},\boldsymbol{\gamma}),t), \tag{5.2}$$

where G^* is the Euler-Poincare dual of G and W_m^* is the weight system that derives from exchanging the order of α_v and β_v in the weight system at each vertex (*i.e.* take $\Gamma = \delta(E(G))$ in Definition 5.5). This led to a new duality result for the topological Tutte polynomial, also in [33, 34]. We are now able to extend

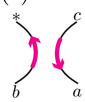
this Euler-Poincare duality to a full twisted duality property for the topological transition polynomial. This twisted duality relation says that the topological transition polynomial of the medial graph of G is the same as that of the medial graph of any of the twisted duals, provided the weight system is appropriately permuted. We will apply this twisted duality relation in the subsequent sections to derive new duality properties for the Penrose and topological Tutte polynomials.

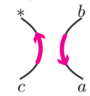
Theorem 5.6. *Let G be an embedded graph with embedded medial graph G_m, and let $\Gamma = \prod_{i=1}^{6} \xi_i(A_i)$ where the A_i's partition $E(G)$, and the ξ_i's are the six elements of $\mathfrak{G}$. Then,*

$$q\left(G_m; W_m, t\right) = q\left(G_m^{\Gamma}; W_m^{\Gamma}, t\right),$$

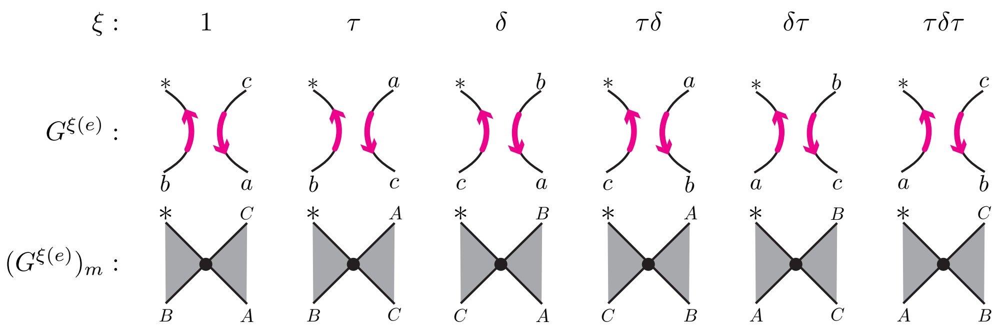
or equivalently,

$$Q(G; (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), t) = Q(G^{\Gamma}, (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})^{\Gamma}, t).$$

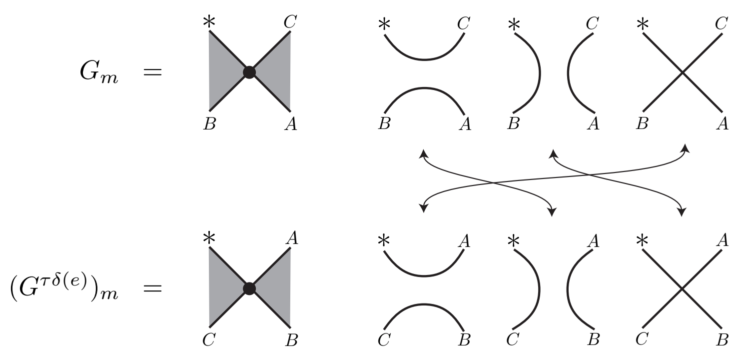
Proof. It suffices to prove this for one edge at a time, *i.e.* to prove the property for $G^{\xi(e)}$ where $\xi \in (G)$. The effect of $\xi(e)$ on the medial graph is shown in the table below. The first row gives ξ. The second row shows the effect on the arrow presentation of the arrows labelled e in the disc presentation of $G^{\xi(e)}$. The labels $*, a, b, c$ are the labels on the disc(s) on either side of e, and the configurations of the discs are otherwise identical (so that, for example, if 

represents two discs, then 

represents one disc, and vice versa). The third row shows the changes in the cyclic order of the vertices about v_e in $(G^{\xi(e)})_m$.



Note that if we give the labels a, b, c the order (a, b, c), then they are permuted in the second row of the table according to $\eta(\xi)$, and hence $(\alpha_{v_e}, \beta_{v_e}, \gamma_{v_e})$ are permuted as claimed. We illustrate this for $G^{\tau\delta(e)}$ below, leaving the other cases to the reader.



□

Note that by taking take $\Gamma = \delta(E(G))$, the result of Equation 5.1 is now just an immediate corollary of Theorem 5.6.

Corollary 5.7. *$Q(G; (\mathbf{1}, \mathbf{1}, \mathbf{1}), t)$ is constant on $Orb(G)$, where $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ is the weight system that assigns a 1 to every transition.*

Proof. This follows from Theorem 5.6 since $(\mathbf{1}, \mathbf{1}, \mathbf{1}) = (\mathbf{1}, \mathbf{1}, \mathbf{1})^{\Gamma}$ for all Γ. □

Note that $Q(G;(\mathbf{1},\mathbf{1},\mathbf{1}),t)$ is just the generating function for the number of Eulerian k-partitions (see [61]) of G_m. It is an open question to characterize graphs such that $Q(G;(\mathbf{1},\mathbf{1},\mathbf{1}),t) = Q(H;(\mathbf{1},\mathbf{1},\mathbf{1}),t)$ when G and H are not twisted duals of one another.

We emphasize that in the following theorem the contraction of a loop e (or any other edge e for that matter) is defined by $G/e := G^{\delta(e)} - e$.

Theorem 5.8. *Let G be an embedded graph and $e \in E(G)$. Then*

$$Q(G;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) = \alpha_e Q(G/e;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) + \beta_e Q(G-e;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) + \gamma_e Q(G^{\tau(e)}/e;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t).$$

Proof. The identity follows from Propositions 4.31 and 5.3 upon observing that twisting an edge of a medial graph does not change the number of cycles in a transition state. ☐

Corollary 5.9. *Let G be an embedded graph, $e \in E(G)$ and $\xi \in \mathfrak{G}$. Further, let $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})^{\xi(e)} = (\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}')$. Then*

$$Q(G;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) = \alpha'_e Q(G^{\xi(e)}/e;(\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'),t) + \beta'_e Q(G^{\xi(e)}-e;(\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'),t) + \gamma'_e Q(G^{\tau\xi(e)}/e;(\boldsymbol{\alpha}',\boldsymbol{\beta}',\boldsymbol{\gamma}'),t).$$

Proof. The identity follow easily from Theorems 5.8 and 5.6. ☐

6. The Penrose polynomial

6.1. **The Penrose polynomial for embedded graphs.** We apply the results of Section 5 to the Penrose polynomial, $P(G,\lambda)$. This polynomial graph invariant for plane graphs was originally defined implicitly by Penrose [71] in the context of tensor diagrams in physics, but turned out to have remarkable graph theoretic properties. Excellent graph theoretical expositions are given by Aigner in [2, 3, 4], with additional exploration of its properties by Aigner and Mielke [5], Ellis-Monaghan and Sarmiento [27], Sarmiento [72], and Szegedy [77]. In this section we use the generalized transition polynomial to define a "topological" Penrose polynomial that extends the original the Penrose polynomial to embedded graphs, much as Bollobás and Riordan have extended the Tutte polynomial to embedded graphs in [9, 10].

Given its origins, the Penrose polynomial has some surprising properties, particularly with respect to graph colouring. The Four Color Theorem is equivalent to showing that every plane, cubic, connected graph can be properly edge-coloured with three colours. The Penrose polynomial, when applied to plane, cubic, connected graphs, encodes exactly this information (see Penrose [71]): if G is a plane, cubic, connected graph, then

$$P(G;3) = \left(\frac{-1}{4}\right)^{\frac{v(G)}{2}} P(G;-2) = \text{the number of edge 3-colourings of } G. \tag{6.1}$$

One of our goals of this section is to determine properties of the Penrose polynomial that can be extended to embedded graphs. We give counter examples for some of those that can not, and new results, including colouring results, for a number that can. We also use the ribbon group action to find new identities for the Penrose polynomial on embedded graphs. (We will not continue to use the adjective 'topological', instead referring to both the original polynomial and the extension defined here simply as the Penrose polynomial.)

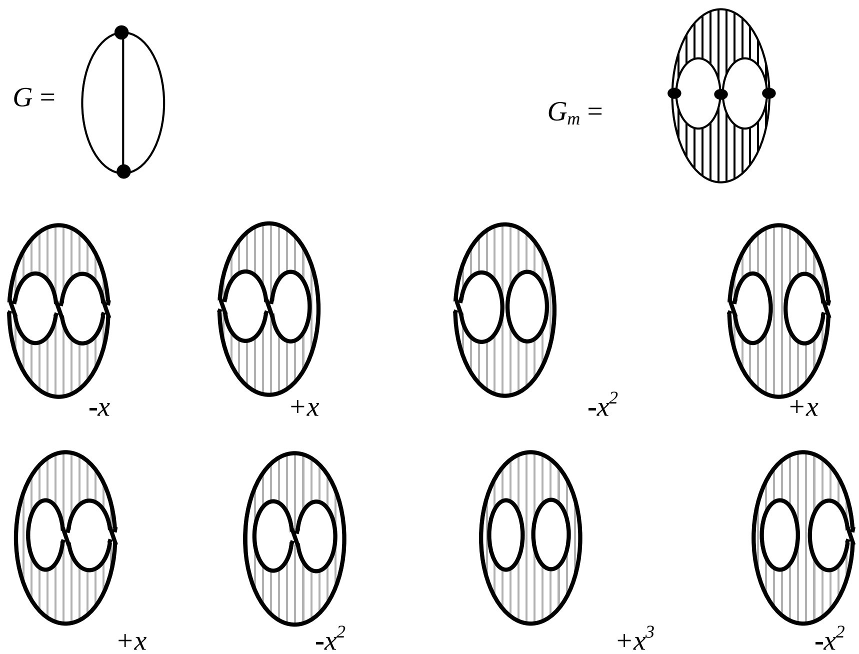
Like the generalized transition polynomial, of which it is a specialization, the original Penrose polynomial of a graph G can be computed using either a state model formulation or a linear recursion relations applied to its medial graph (see Jaeger [45] for example). We will use both approaches here.

Definition 6.1 (The original Penrose polynomial of a plane graph)**.** Let G be a plane graph with cannonically colored medial graph G_m, let $St(G_m)$ be the set of states of G_m, and let $St'(G_m)$ be the set of states with no black splits. Let W_P be the medial weight system with $\alpha_v = 1$, $\beta_v = 0$, and $\gamma_v = -1$ for all $v \in V(G_m)$. Then,

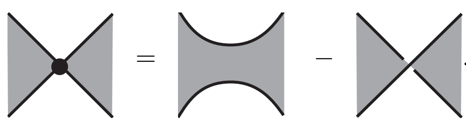
$$P(G;x) = \sum_{s \in St(G_m)} \omega_P(s) x^{c(s)} = \sum_{s \in St'(G_m)} \left((-1)^{cr(s)} x^{c(s)} \right),$$

where $c(s)$ is the number of components in the graph state s, and $cr(s)$ is the number of crossing vertex states in the state s.

Example 6.2. As an example, if G is the plane θ-graph consisting of two vertices joined by three edges in parallel, then $P(G;x) = x^3 - 3x^2 + 2x$, as shown below.



The Penrose polynomial may also be computed via a linear recursion relation (see Jaeger [45] for example), by repeated applying the skein relation



to any vertex of degree 4 in G_m, and at the end, evaluating each of the resulting cycles to x.

This immediately suggests how the Penrose polynomial can be extended to embedded graphs via the topological transition polynomial. Definition 6.3 generalizes the relation between the original Penrose polynomial and the transition polynomial given by Jaeger [45] and Ellis-Monaghan and Sarmiento [28, 27], where in the latter properties of the generalized transition polynomial are used to derive enumeration formulae for the original Penrose polynomials. Here we will similarly use properties determined in the Section 5 for the topological transition polynomial to find new identities for the Penrose polynomial on embedded graphs.

Definition 6.3. Let G be an embedded graph and G_m be its canonically checkerboard coloured embedded medial graph. The *Penrose weight system* $W_P(G_m)$ is defined by letting $\alpha_v = 1$, $\beta_v = 0$, and $\gamma_v = -1$ for all $v \in V(G_m)$ in the medial weight system. We will also denote this weight system by $(\mathbf{1}, \mathbf{0}, -\mathbf{1})$.

The *Penrose polynomial* $P(G; \lambda) \in \mathbb{Z}[\lambda]$ of an embedded graph G is then

$$P(G; \lambda) = q(G_m; W_P, \lambda) = Q(G; (\mathbf{1}, \mathbf{0}, -\mathbf{1}), \lambda),$$

where Q is the topological transition polynomial.

We will define the set of *Penrose states*, $\mathcal{P}(G)$, of a ribbon graph G to be the set of all graph states such that each vertex state is either a white split or a crossing. The Penrose polynomial can then be expressed as the state sum

$$P(G; \lambda) = \sum_{s \in \mathcal{P}(G)} (-1)^{cr(s)} \lambda^{c(s)}. \tag{6.2}$$

We can use this state sum to express the Penrose polynomial of a ribbon graph G in terms of twisted duals of G, as opposed to an expression in terms of its medial graph G_m.

Proposition 6.4. *Let G be a ribbon graph, then*

$$P(G;\lambda) = \sum_{A\subseteq E(G)} (-1)^{|A|}\lambda^{f(G^{\tau(A)})},$$

recalling that $f(G)$ is the number of faces of G.

Proof. This is not hard to see by looking at Example 6.2 and considering the facial walks of $G^{\tau(A)}$. It may be proved formally by noting that there is a one-to-one correspondence between the subsets $A \subseteq E(G)$ and flat arrow marked Penrose states of G_m by taking a flat crossing v_e in $\vec{s_A}$ whenever $e \in A$, and taking a flat white split else. Clearly $cr(s_A) = |A|$. If $\vec{b}$ is the state of G_m consisting of all flat black splits, so $G = (G_m)_{\vec{b}}$, then the flat arrow marked Penrose state corresponding to $A \subseteq E(G)$ can be written as $\vec{b}^{\delta(E-A)\delta\tau(A)}$. We then have

$$(G^{\tau(A)})^* = (((G_m)_{\vec{b}})^{\tau(A)})^* = (G_m)_{\vec{b}^{\delta(E-A)\delta\tau(A)}},$$

where the second equality follows from Proposition 4.27.

Note that f F is a 4-regular embedded graph, s is a state of F, and $\vec{s}$ any arrow marking of s, then $f((F_{\vec{s}})^*) = c(\vec{s}) = c(s)$, since $f((F_{\vec{s}})^*) = v(F_{\vec{s}}) = c(\vec{s}) = c(s)$.

Thus, since $G^{\tau(A)} = ((G^{\tau(A)})^*)^*$,

$$f(G^{\tau(A)}) = f(((G^{\tau(A)})^*)^*) = f(((G_m)_{\vec{b}^{\delta(E-A)\delta\tau(A)}})^*) = c(\vec{b}^{\delta(E-A)\delta\tau(A)}) = c(s_A).$$

The result then follows. $\square$

This expression of the Penrose polynomial in terms of the set of twists of a ribbon graph will be convenient later.

6.2. **Twisted duality and identities for the Penrose polynomial.** One of the major advantages of considering the topological Penrose polynomial is that it satisfies various identities that are not realizable in terms of plane graphs. Many of these identities arise by considering twisted duality. For example, we see in this context that unlike the classical Penrose polynomial, the topological Penrose polynomial has deletion-contraction reductions similar to those of the Tutte polynomial.

Proposition 6.5. *The Penrose polynomial of an embedded graph G has the following properties.*

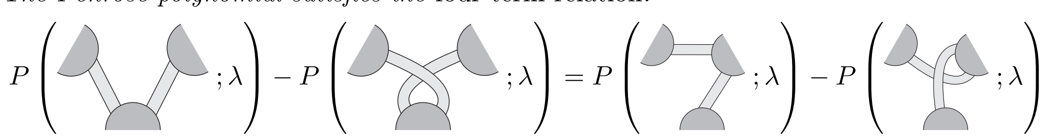
(i) If $A \subset E(G)$ then $P(G;\lambda) = (-1)^{|A|}P(G^{\tau(A)};\lambda)$, and in particular $|P(G;\lambda)|$ is an invariant of the orbits of the twist, i.e. *of combinatorial maps. Furthermore, $P(G;\lambda) = (-1)^{|A|}Q(G,(\mathbf{1},\mathbf{0},-\mathbf{1})^{\tau(A)},\lambda)$.*

(ii) If $e \in E(G)$, then

$$P(G;\lambda) = P(G^{\delta(e)};\lambda) - P(G^{\delta\tau(e)};\lambda) = P(G^{\delta(e)};\lambda) + P(G^{\delta\tau\delta(e)};\lambda).$$

(iii) If e is a non-twisted loop of G that bounds a 2-cell, then $P(G;\lambda) = (\lambda-1)P(G-e;\lambda)$.

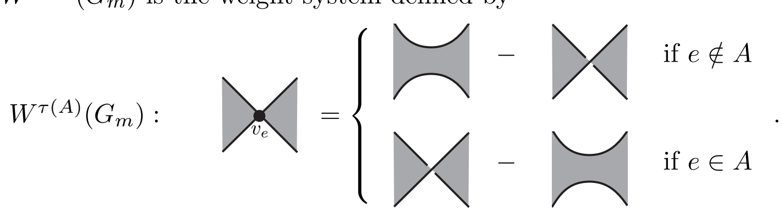
(iv) The Penrose polynomial satisfies the four-term relation*:*



where the four ribbon graphs in the figure are identical except in the region shown.

Proof. We will prove the properties one by one.

(i) Recall that $W^{\tau(A)}(G_m)$ is the weight system defined by



Thus,

$$P(G^{\tau(A)};\lambda)$$

$$= Q(G^{\tau(A)}; (\mathbf{1}, \mathbf{0}, -\mathbf{1}), \lambda)$$
$$= Q((G^{\tau(A)})^{\tau(A)}; (\mathbf{1}, \mathbf{0}, -\mathbf{1})^{\tau(A)}, \lambda)$$
$$= Q(G; (\mathbf{1}, \mathbf{0}, -\mathbf{1})^{\tau(A)}, \lambda)$$

, where the second equality follows from Theorem 5.6. Also,

$$P(G^{\tau(A)}; \lambda)$$
$$= (-1)^{|A|} Q(G; (\mathbf{1}, \mathbf{0}, -\mathbf{1}), \lambda)$$

,

since for each Penrose state s of G_m we have that $(-1)^{|A|}\omega_P(s) = \omega_P{}^{\tau(A)}(s)$. This proves both parts of Item (i).

(ii) We begin with the equation

$$q(G_m; W_P, \lambda) = q(G_m; W_P^{\delta(e)}, \lambda) - q(G_m; W_P^{\tau\delta(e)}, \lambda). \tag{6.3}$$

To see why this equation holds, we note that by applying Proposition 5.3 to the weight systems W_P, $W_p^{\delta(e)}$ and $W_p^{\tau\delta(e)}$ respectively we have

$$q(G_m; W_p, \lambda) = q((G_m)_{wh(v_e)}; W_p, \lambda) - q((G_m)_{cr(v_e)}; W_p, \lambda),$$
$$q(G_m; W_p^{\delta(e)}, \lambda) = q((G_m)_{bl(v_e)}; W_p^{\delta(e)}, \lambda) - q((G_m)_{cr(v_e)}; W_p^{\delta(e)}, \lambda)$$

and

$$q(G_m; W_p^{\tau\delta(e)}, \lambda) = -q(G_m(G_m)_{wh(v_e)}; W_p^{\tau\delta(e)}, \lambda) + q((G_m)_{bl(v_e)}; W_p^{\tau\delta(e)}, \lambda).$$

Substituting the three identities above into the left and right hand sides of Equation 6.3 will give the required equality.

We will now express each of the terms in Equation 6.3 in terms of the Penrose polynomial. By definition, we have

$$P(G; \lambda) = q(G_m; W_P, \lambda). \tag{6.4}$$

For the second term we have

$$\begin{aligned} P(G^{\delta(e)}; \lambda) &= q((G^{\delta(e)})_m; W_P, \lambda) \\ &= q(((G^{\delta(e)})^{\delta(e)})_m; W_P^{\delta(e)}, \lambda) \\ &= q(G_m; W_P^{\delta(e)}(G_m), \lambda), \end{aligned} \tag{6.5}$$

where the first equality follows by definition, the second follows from Theorem 5.6, and the third follows from the fact that $(\delta(e))(\delta(e)) = 1(e)$.

Finally, we can rewrite third term of Equation 6.3 as follows.

$$\begin{aligned} P(G^{\delta\tau(e)}; \lambda) &= q((G^{\delta\tau(e)})_m; W_P, \lambda) \\ &= q(((G^{\delta\tau(e)})^{\tau\delta(e)})_m; W_P^{\tau\delta(e)}, \lambda) \\ &= q(G_m; W_P^{\tau\delta(e)}, \lambda), \end{aligned} \tag{6.6}$$

where the first equality follows by definition, the second follows from Theorem 5.6, and the third follows from the fact that $(\delta\tau(e))(\tau\delta(e)) = 1(e)$.

The result stated in the proposition then follows by substituting the identities in Equations 6.4, 6.5 and 6.6 into Equation 6.3.

(iii) The proof of this property is straight forward and is therefore omitted.

(iv) Let G_i, for $i = 1, \ldots, 4$, be the four embedded graphs shown in the four-term relation in the order shown in the defining figure. There is a canonical bijection between their edge sets. Identify the edge sets of each of the G_i using this correspondence. Let e and f denote the distinguished edges of the G_i. Then since by Proposition 6.4

$$P(G_i; \lambda) = \sum_{\substack{A \subseteq E(G_i) - \{e,f\} \\ B \subseteq \{e,f\}}} (-1)^{|A \cup B|} \lambda^{f(G_i^{\tau(A \cup B)})},$$

it is enough to show that for a fixed subset A of $E(G_i) - \{e, f\}$, we have

$$\sum_{B\subseteq\{e,f\}} (-1)^{|B|}\lambda^{f(G_1^{\tau(A\cup B)})} - \sum_{B\subseteq\{e,f\}} (-1)^{|B|}\lambda^{f(G_2^{\tau(A\cup B)})} - \sum_{B\subseteq\{e,f\}} (-1)^{|B|}\lambda^{f(G_3^{\tau(A\cup B)})} + \sum_{B\subseteq\{e,f\}} (-1)^{|B|}\lambda^{f(G_4^{\tau(A\cup B)})} = 0.$$

This identity is easily verified by calculation.

□

The fact that the state sum defining the Penrose polynomial satisfies the four-term relation is a well known fact in the theory of Vassiliev invariants of knots. This is since the Penrose polynomial can be expressed in term of the $\mathfrak{so}_N$ weight system.

By moving into the broader class of embedded graphs, we now see that the Penrose polynomial actually does have deletion/contraction reductions, like those for the Tutte polynomial, albeit with a slight twist.

Theorem 6.6. *Suppose G is an embedded graph, and $e \in E(G)$. Then*

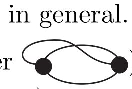
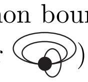
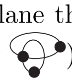
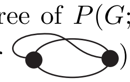
(i)
$$P(G;\lambda) = P(G/e;\lambda) - P(G^{\tau(e)}/e;\lambda);$$
(ii)
$$P(G;\lambda) = P(G/e;\lambda) - P(G^{\tau\delta(e)}/e;\lambda);$$
(iii)
$$P(G;\lambda) = P(G^{\tau\delta(e)}/e;\lambda) - P(G^{\tau\delta(e)} - e;\lambda),$$
or equivalently,
$$P(G^{\delta\tau(e)};\lambda) = P(G/e;\lambda) - P(G - e;\lambda).$$

Proof. Items (i) and (ii) follow immediately from Theorem 5.8 and Corollary 5.9 and the definition of contraction. (Item (ii) also follows from (i) since $G^{\tau\delta(e)}/e = G^{\delta\tau\delta(e)} - e = G^{\tau\delta\tau(e)} - e = G^{\delta\tau(e)} - e = G^{\tau(e)}/e$.) For Item (iii), apply Item (i) to $G^{\delta(e)}$, then add and subtract $P(G^{\delta(e)} - e;\lambda)$ and use Item (ii) to get that $P(G;\lambda) - P(G^{\delta(e)};\lambda) = P(G - e;\lambda) - P(G/e;\lambda)$. The result then follows from Proposition 6.5 Item 2.

□

6.3. **The Penrose polynomial and colourings.** The Penrose polynomial of a plane graph is known to satisfy several combinatorial identities and has numerous connections with graph colouring. It is natural to ask which of these properties hold off of the plane. Here we will discuss relations between the Penrose polynomial for non-plane graphs and graph colouring. Before moving on to the discussion of graph colouring, we observe that many of the basic properties of the Penrose polynomial of a plane graph given by Penrose in [71] and Aigner in [2] do not hold for non-plane graphs.

For example, the follow properties proved by Aigner in [2] do not hold for embedded graphs in general.

(i) If G is plane and Eulerian then $P(G;2) = 2^{v(G)}$. This does not hold in general (consider ).
(ii) If G is plane and has two regions with a common boundary e, then $P(G;\lambda) = 2P(G/e;\lambda)$. This is not true for general embedded graphs (consider ).
(iii) If G is plane then the leading term of $P(G;\lambda)$ is 1. This is not true for general embedded graphs (consider ).
(iv) If G is plane then the degree of $P(G;\lambda)$ is the number of faces of G. This is not true for general embedded graphs (consider ).
(v) If G is plane and Eulerian then $P(G;-1) = 2^{e(G)}$. This does not hold in general (consider ). (Although $|P(G;-1)| < 2^{e(G)}$ for all G.)

In [2], Aigner proved the following theorem which relates the number of proper k colourings of a graph and the Penrose polynomial.

Theorem 6.7. *Let G be a plane graph, then for all $k \in \mathbb{N}$ we have*

$$\chi(G^*; k) \leq P(G; k).$$

In Theorem 6.10, we will complete Theorem 6.7 by showing that the Penrose polynomial of plane graph G is in fact *equal* to a sum of specific chromatic polynomials. Moreover, this sum is indexed by the orbit of the twist-action of G. It turns out that the expression $\chi(G^*; k)$ is a single summand in our expression for the Penrose polynomial $P(G; k)$. Theorem 6.7 then follows from Theorem 6.10 as a corollary.

We will need the concept of an admissible k-valuation from [2].

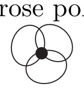
Definition 6.8. Let $G = (V(G), E(G))$ be an embedded graph and G_m be its medial ribbon graph. A *k-valuation* of G_m is a k-edge colouring $\phi : E(G_M) \to \{1, 2, \ldots, k\}$ such that for each i and every vertex v_e of G_m, the number of i-coloured edges incident with v_e is even.

A k-valuation is said to be *admissible* if at each vertex of G_m the k-valuation is of one of the following two types:

where $i \neq j$. The two local configurations above correspond to *white split states* and *crossing states* respectively.

The following theorem, which expresses the Penrose polynomial of a plane graph in terms of k-valuations, is due to Jaeger ([44] Proposition 13, see also [2] Proposition 4).

Theorem 6.9. *If G is a plane graph, then for each $k \in \mathbb{N}$, $P(G; k)$ is equal to the number of admissible k-valuations of the medial ribbon graph G_m of G.*

This characterization of the Penrose polynomial does not extend to a general embedded graph, however. For example, the embedded graph  has Penrose polynomial $-\lambda^3 + 4\lambda^2 - 3\lambda$, but has $k^3 - 2k^2 - k$ admissible k-valuations, for $k \geq 3$.

We are now in a position to be able to state and prove our generalization of Aigner's inequality $\chi(G^*; k) \leq P(G; k)$ which relates the chromatic and Penrose polynomials.

Theorem 6.10. *If $G = (V(G), E(G))$ is a plane graph, then*

$$P(G; \lambda) = \sum_{A \subseteq E(G)} \chi((G^{\tau(A)})^*; \lambda).$$

Proof. If H is a ribbon graph, we define a *boundary k-valuation* φ to be a k-colouring $\varphi : \{1, 2, \ldots, k\} \to \mathrm{BC}(H)$ of the boundary components of H such that if two boundary components share a common edge then they are assigned different colours. This corresponds to a proper face-colouring of H when H is viewed as cellularly embedded, and hence to a proper coloring of H^*. Thus, the number of boundary k-valuations of a ribbon graph H is equal to $\chi(H^*; k)$, for $k \in \mathbb{N}$.

Each admissible k-valuation of G_m induces a Penrose state of G_m by assigning a crossing vertex state of G_m if there is a crossing state at the vertex in the admissible k-valuation, and assigning a white split state otherwise.

Note from Figure 4 that the edges of G_m follow precisely the boundaries of G if we choose a crossing state at vertices of G_m corresponding to twisted edges of G and a white split state at vertices corresponding to untwisted edges of G. Thus an admissible edge coloring of G_m corresponds exactly to a boundary k-valuation of $G^{\tau(A)}$, where A is the set of edges corresponding to vertices of G_m where the local coloring configuration of the incident edges from the admissible k-valuation gives a crossing state.

If G is a plane graphs, then by Theorem 6.9 $P(G; k)$ is equal to the number of admissible k-valuationsfor all $k \in \mathbb{N}$. This is equal to the sum over all Penrose states of admissible k-valuations which induce the given

Penrose state. Thus,

$$\sum_{s\in\mathcal{P}(G_m)} (\text{number of admissible } k\text{-valuations inducing } s)$$

$$= \sum_{A\subseteq E(G)} (\text{number of boundary } k\text{-valuations of } G^{\tau(A)})$$

$$= \sum_{A\subseteq E(G)} \chi((G^{\tau(A)})^*; k).$$

Since this holds for all natural numbers k, it follows that the polynomials $P(G;\lambda)$ and $\sum_{A\subseteq E(G)} \chi((G^{\tau(A)})^*;\lambda)$ are equal, as required.

□

Theorem 6.10 can be used to reformulate the four color theorem.

Corollary 6.11. *The following statements are equivalent:*

(i) the four colour theorem is true;
(ii) for every connected, bridgeless plane graph G there exits $A \subseteq E(G)$ such that $\chi((G^{\tau(A)})^*;3) \neq 0$;
(iii) for every connected, bridgeless plane graph G there exits $A \subseteq E(G)$ such that $\chi((G^{\tau(A)})^*;4) \neq 0$;

Proof. Corollary 9 of [2] states that that four colour theorem is equivalent to showing that $P(G;3) > 0$ or $P(G;4) > 0$ for all connected, bridgeless plane graphs G. Since $\chi(G';k) \geq 0$ for all $k \in \mathbb{N}$ and graphs G', Theorem 6.10 tells us that $P(G;k) > 0$ if and only if one of the summands $\chi((G^{\tau(A)})^*;k) \neq 0$. The result then follows. □

7. The topochromatic polynomial

Extending the classical Tutte polynomial to a topological Tutte polynomial for embedded graphs was first explored by Las Vergnas [59] in the language of combinatorial geometries, and then by Bollobás and Riordan [9, 10] in the context of ribbon graphs. A multivariate generalization, the topochromatic polynomial, was introduced in [65] In this section we will show that the behaviour of the topological transition polynomial under the twisted duality operation from Section 5 provides a framework for understanding the behaviour of the topological Tutte polynomial under partial duality. The behaviour of the topological Tutte polynomial under partial duality is significant because of its knot theoretical applications (since partial duality intertwines various recent realizations of the Jones polynomial of a (virtual) link (see [46, 52]) as evaluations of the topological Tutte polynomial). We will begin by giving a brief outline of the role that partial duality plays in knot theory.

Seminal results of Thistlethwaite [78] and Kauffman [50] that relate the Tutte, dichromatic and Jones polynomials were extended in two different ways by Chmutov and Pak in [21] and Dasbach et. al. in [23]. In [21] it was shown that the Jones polynomial of checkerboard colourable virtual link is an evaluation of the topochromatic polynomial of its signed Tait graph. Using a different construction, in [23] it was shown that the Jones polynomial is an evaluation of the topological Tutte polynomial of an associated (unsigned) embedded graph. Although both of these results generalize Thistlethwaite's connection between the Tutte and Jones polynomials, they do so in very different ways. In [66], the first author unified these relations between graph and knot polynomials through an "unsigning" procedure on the Tait graph of a link. This "unsigning" is a special case of the subsequently defined partial duality operation on embedded graphs. A third extension of Thistlethwaite's result was given by Chmutov and Voltz in [22], where the Jones polynomial of an arbitrary virtual link was shown to be an evaluation of the topochromatic polynomial. In [17], Chmutov introduced the partial duality operation and studied the behaviour of the (2-variable) topochromatic polynomial under this operation. He then went on to show how this behaviour under partial duality provides a framework which unifies all of these new relations between knot and graph polynomials.

In this section we will explain how twisted duality and the transition polynomial provide a natural, unified framework for these recently discovered connections between graph theory and knot theory. We will show that a partial duality relation for the topochromatic polynomial arises naturally from the transition polynomial.

We will also show that Vignes-Tourneret's duality relation for the multivariate signed Bollobás-Riordan polynomial from [86] follows from our relation.

7.1. **The topochromatic polynomial.** We first establish some notation. The notion of a spanning ribbon subgraph is clear, and we define a spanning subgraph of an embedded graph to be the embedded graph corresponding to a spanning ribbon subgraph. (This ensures that the spanning subgraphs of a cellularly embedded graph are also cellularly embedded, although not necessarily in the same surface as the original graph.) For a embedded graph G, we let $\kappa(G), r(G)$, and $n(G)$ be, respectively, the number of connected components, rank, and nullity of the underlying abstract graph. If G is viewed as a ribbon graph, then $f(G)$ is the number of boundary components of the surface defining the ribbon graph. Also, the function t records orientability of an embedded graph G by $t(G) = 0$ if G is embedded in an orientable surface (or equivalently, the ribbon graph is an orientable surface) and $t(G) = 1$ if G is embedded in a non-orientable surface (or equivalently, the ribbon graph is a non-orientable surface). Finally, $g(G)$ is the genus of G, or the sum of the genuses of the maximal components of G if G is not connected. When $A \subseteq E(G)$, then $r(A)$, $\kappa(A)$, $n(A)$, $f(A)$, $t(A)$, and $g(A)$ each refer to the spanning subgraph of G on the edge set A, with embedding inherited from G.

The topological Tutte polynomial of Las Vergnas [59] was first defined in terms of the combinatorial geometry of an embedded graph (*i.e.* its circuit matroid), so we provide some translation of terminology before stating the definition. The rank functions are given in the context of $B(G)$ and $C(G)$, that is, the bond and circuit geometries of G, but we may translate this to graph theoretic terms by noting that $r(C(G)) = r(G)$, $r(B(G)) = n(G)$, $r(C(G^*)) = v(G^*) - k(G)$, $r(B(G^*)) = n(G^*) = r(G) + 2g(G)$. When $A \subseteq E(G)$, we have $r_{C(G^*)}(A) = r(A)$, $r_{B(G)}(A) = n(A)$, $r_{B(G^*)}(A) = r(A) + 2g(A)$. With this, we have the folloing defintion:

Definition 7.1. (Las Vergnas [59]) Let G be an embedded graph. Then

$$L(G; x, y, z) = \sum_{A \subseteq E(G)} (x-1)^{r(C(G)) - r_{C(G)}(A)} (y-1)^{|A| - r_{B(G^*)}(A)} z^{r(B(G^*)) - r(C(G)) - (r_{B(G^*)}(A) - r_{C(G)}(A))}.$$

There are two definitions of the topological Tutte polynomial, a generating function (or subset expansion) formulation and a linear recursion formulation, that were shown to be equivalent by Bollobás and Riordan [10]. We will use the subset expansion formulation.

Definition 7.2. (Bollobás and Riordan [10]) Let G be a ribbon graph. Then

$$\begin{aligned} R(G; x, y, z, w) = & \sum_{A \subseteq E(G)} (x-1)^{r(G) - r(A)} y^{n(A)} z^{\kappa(A) - f(A) + n(A)} w^{t(A)} \\ & \in Z[x, y, z, w] / \langle w^2 - w \rangle. \end{aligned}$$

A little computation yields the following translation between the topological Tutte polynomial of Las Vergnas and that of Bollobás and Riordan [10].

Proposition 7.3.

$$y^{g(G)} z^{-g(G)} L(G; x, y+1, \frac{1}{yz}) = R(G; x, y, z, 1).$$

The *topochromatic polynomial* is simply a shift, with addition of weights, of these topological Tutte polynomials, as follows.

Definition 7.4. (Moffatt [65] Let G be an embedded graph. Let a, c, and w be indeterminates, and let $\boldsymbol{b} := \{b_e | e \in E(G)\}$ be a set of indeterminates indexed by $E(G)$. The *topochromatic* polynomial is

$$Z(G; a, \boldsymbol{b}, c, w) = \sum_{H \subseteq G} a^{k(H)} \left(\prod_{e \in E(H)} b_e \right) c^{f(H)} w^{t(H)} \in \mathbb{Z}[a, \boldsymbol{b}, c, w] / \langle w^2 - w \rangle,$$

where the sum is over all (embedded) spanning subgraphs H of G.

The topochromatic polynomial is a shift of the topological Tutte polynomials, just as the dichromatic polynomial $Z(G;a,b) := \sum_{H\subseteq G} a^{k(H)}b^{e(H)}$ is a shift of the Tutte polynomial:

$$T(G;x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)}Z(G;(x-1)(y-1),(y-1)).$$

It is for this reason that we use the name "topochromatic polynomial" here. The topochromatic polynomial was first defined in [65] as a multivariate generalization of the Bollobás and Riordan's topological Tutte polynomial from [9, 10]. The generalization is in the spirit of the extensions of the Tutte polynomial to edge weighted graphs from [88], [8] and [75]. The introduction of the topochromatic polynomial was necessitated by some of the knot theoretical applications in considered in [65]. The topochromatic polynomial has since found other applications, such as in [40], where it was used in to study the behaviour of the topological Tutte polynomial under the 2-sum operation.

In the literature the topochromatic polynomial is usually applied to the ribbon graph realization of an embedded graph. Here we will generally avoid fixing a realization of an embedded graph.

The topochromatic polynomial contains as specializations both the topological Tutte polynomial of Las Vergnas and that of Bollobás and Riordan, and also the classical Tutte polynomial and the restricted normal form of the multivariate Tutte polynomial (see [65] for details). We will see below that it is also equivalent to the signed multivariate Bollobás-Riordan polynomial from Vignes-Tourneret [86].

It was observed in [40] that the topochromatic polynomial satisfies a deletion contraction relation

$$Z(G;a,\boldsymbol{b},c,w) = Z(G-e;a,\boldsymbol{b}',c,w) + b_e Z(G/e;a,\boldsymbol{b}',c,w)$$

where $e\in E(G)$ is a non-loop edge and $\boldsymbol{b}' = \boldsymbol{b} - \{b_e\}$. This deletion-contraction relation allows one the calculation of the topochromatic in terms of bouquets.

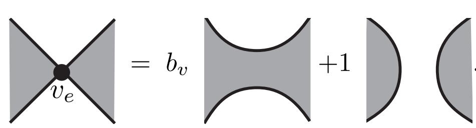
If e and $\delta(e)$ are non-loop edges, then we have that $Z(G-e;a,\boldsymbol{b}',c,w) = Z((G^{\delta(e)})/e;a,\boldsymbol{b}',c,w)$ and so we obtain a dual-contraction relation for the topochromatic polynomial:

$$Z(G;a,\boldsymbol{b},c,w) = Z((G^{\delta(e)})/e;a,\boldsymbol{b}',c,w) + b_e Z(G/e;a,\boldsymbol{b}',c,w)$$

The classical Tutte polynomial, $T(G;x,y)$, among many other properties, encodes information about families of Eulerian circuits in the medial graph of a plane graph. This theory is the result of a relation between the classical Tutte polynomial and the Martin, or circuit partition, polynomial. In [34] this theory was extended to ribbon graphs, giving an analogous result relating the topological Tutte polynomial of Bollobás and Riordan [9, 10] for a ribbon graph to the transition polynomial of its topological medial graph, noting that the transition polynomial of [28] is a multivariate generalization of the circuit partition polynomial. The original relation between the Tutte polynomial and the Martin polynomial can be found in Martin's 1977 thesis [63], with the theory considerably extended by Martin [64], Las Vernas [60, 61, 62], Jaeger [45], Bollobás [11], and [26, 29, 30, 28]. An overview can be found in Ellis-Monaghan and Merino [31, 32]. These ideas can be extended to the topochromatic polynomial, as was done for full duality in the unweighted case in [59, 34, 33], and see also [65, 67] and Chmutov [17]. Here we extend them to multivariate generalizations and partial duality. We are particularly interested in an extension of the relation between the topological Tutte polynomial and the transition polynomial which will allow us to apply our previous results on twisted duality to the topochromatic polynomial.

Let G be an embedded graph and G_m be its embedded medial graph equipped with the canonical checkerboard colouring. We define the weight system $W_Z(G_m)$ by

$W_Z(G_m)$:



The following proposition expresses the topochromatic polynomial and the transition polynomial using the weight system W_Z.

Proposition 7.5. *If G is an embedded graph and G_m is its embedded medial graph, then*

$$Q(G;(\boldsymbol{b},\boldsymbol{1},\boldsymbol{0}),c) = Z(G;1,\boldsymbol{b},c,1).$$

Proof. By definition,

$$Q\left(G;(\boldsymbol{b},\mathbf{1},\mathbf{0}),c\right)=\sum_{s}\omega_{Z}(s)c^{c(s)}=\sum_{s}\left(\prod_{v_e\in Wh(s)}b_e\right)c^{c(s)},$$

where the sum is over all graph states s with no crossing states and where $Wh(s)$ is the set vertices with white split states in the graph state s.

We can define a bijection between the set of embedded spanning subgraphs of G and the set of graph states of G_m by associating an embedded spanning subgraph H_s of G by setting $e \in H_s$ if and only if the vertex state $v_e \in Wh(s)$. It is then clear that, for every graph state, $c(s) = f(H_s)$. By using this bijection, we have

$$Q\left(G;(\boldsymbol{b},\mathbf{1},\mathbf{0}),c\right)=\sum_{s}\left(\prod_{v_e\in Wh(s)}b_e\right)c^{c(s)}=\sum_{H\subseteq G}\left(\prod_{e\in E(H)}b_e\right)c^{f(H)}=Z(G;1,\boldsymbol{b},c,1).$$

□

7.2. **The Penrose and topological Tutte polynomials.** The realization of the topochromatic polynomial as an evaluation of the transition polynomial now allows us to express the topological Penrose polynomial as an evaluation of the topochromatic polynomial. We will use this connection between the two polynomials to find some new combinatorial interpretations of evaluations of the topological Tutte polynomial. We will also express the four colour theorem in terms of the topological Tutte polynomial.

Theorem 7.6. *Let G be an embedded graph and let $H = G^{\tau\delta(E(G))}$, then*

$$P(G;\lambda) = Z(H;1,\mathbf{1},\lambda,1)$$

Proof. We have

$$P(G;\lambda)=Q\left(G;(\mathbf{1},\mathbf{0},-\mathbf{1}),\lambda\right)=Q\left(G^{\tau\delta(E(G))};(-\mathbf{1},\mathbf{1},\mathbf{0}),\lambda\right)=Z(G^{\tau\delta(E(G))};1,\mathbf{1},\lambda,1),$$

where the first equality follows from the definition of the Penrose polynomial (Definition 6.3), the second follows from Theorem 5.6, and the third equality from Proposition 7.5. □

The topological Tutte polynomials of Bollobás and Riordan ([9, 10]) and of Las Vergnas ([59]) can be recovered from the topochromatic polynomial as follows:

$$R(G;x,y,z,w)=(x-1)^{-k(G)}(yz)^{-v(G)}Z(G;(x-1)yz^2,\boldsymbol{b},z^{-1},w),$$

where the edge weights $\boldsymbol{b}$ are given by setting $b_e = zy$ for each $e \in E(G)$. The above theorem can then be used to express the Penrose polynomial in terms of the topological Tutte polynomial of Bollobás and Riordan as in the following corollary.

Corollary 7.7. *Let G be an embedded graph and let $H = G^{\tau\delta(E(G))}$. Then*

$$P(G;\lambda)=\lambda^{k(H)}R(H;\lambda+1,\lambda,\lambda^{-1},1).$$

We can use this identity and the evaluations of the Penrose polynomial from Section 6 to obtain combinatorial interpretations of some points of the topological Tutte polynomial.

Corollary 7.8. *Let G be an embedded graph with the property that $H := G^{\delta\tau(E(G))}$ is plane. Then*

(i) $3R(G;4,3,\frac{1}{3},1) = (-2)(-4)^{v(H)/2}R(G;-1,-2,-\frac{1}{2},1)$ *is equal to the number of edge 3-colourings of H if in addition H is connected and cubic.*

(ii) $k^{k(G)}R(G;k+1,k,\frac{1}{k},1)$ *is equal to the number of admissible k-valuations of H_m, for $k \in \mathbb{N}$.*

(iii) $R(G;\lambda+1,\lambda,\lambda^{-1},1) = \lambda^{-k(G)}\sum_{A\subseteq E(H)}\chi((H^{\tau(A)})^{*};\lambda)$.

Proof. Item (1) follows from Corollary 7.7 and Equation (6.1). Item (1) follows from Corollary 7.7 and Theorem 6.9 Item (3) follows from Corollary 7.7 and Theorem 6.10. □

We can also use the relation between the Penrose and topological Tutte polynomial to obtain a new formulation of the four colour theorem.

Corollary 7.9. *The following statements are equivalent:*

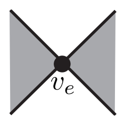
(i) the four colour theorem is true;
(ii) for every connected, loopless plane graph G, $R(G^{\tau(E(G))}; 4, 3, \frac{1}{3}, 1) > 0$;
(iii) for every connected, loopless plane graph G, $R(G^{\tau(E(G))}; 5, 4, \frac{1}{4}, 1) > 0$;

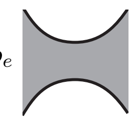
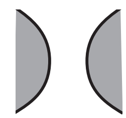
Proof. Corollary 9 of [2] states that that four colour theorem is equivalent to showing that $P(G; 3) > 0$ or $P(G; 4) > 0$ for all connected, bridgeless plane graphs G. By Corollary 7.7, this is equivalent to showing that $3R(G^{\tau\delta(E(G))}; 4, 3, \frac{1}{3}, 1) > 0$ or $4R(G^{\tau\delta(E(G))}; 5, 4, \frac{1}{4}, 1) > 0$, for all connected, bridgeless plane graphs G. Since $G^{\tau\delta(E(G))} = (G^*)^{\tau(E(G))}$, this is equivalent to showing that $R(G^{\tau\delta(E(G))}; 4, 3, \frac{1}{3}, 1) > 0$ or $R(G^{\tau\delta(E(G))}; 5, 4, \frac{1}{4}, 1) > 0$, for all connected, loopless plane graphs G. □

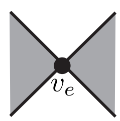
7.3. **Partial duality and the topochromatic polynomial.** In this section we use the relation between the topochromatic and transition polynomials of Theorem 7.5 and the behaviour of the transition polynomial under twisted duality from Theorem 5.6 to find a partial duality relation for the topochromatic polynomial.

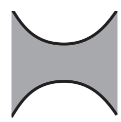
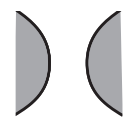
Let G be an embedded graph and let G_m be its embedded medial graph equipped with the canonical checkerboard colouring. Then, from Definition 5.5, for $A \subseteq E(G)$, the weight system $W_Z^{\delta(A)}(G_m)$ is given by reversing the roles of b_e and 1 if e is in A, thus:

$W_Z^{\delta(A)}(G_m)$:

if $e \notin A$ then 

$=$ b_e  $+$ 

if $e \in A$ then 

$=$  $+b_e$  .

Lemma 7.10. *Let G be an embedded graph with embedded medial graph G_m. Then if $A \subseteq E(G)$, we have*

$$Q\left(G;\, (\boldsymbol{b}, \mathbf{1}, \mathbf{0})^{\delta(A)}, c\right) = \left(\prod_{\substack{e\in E(G)\\ e\in A}} b_e\right) Z(G; 1, \boldsymbol{b}_A, c, 1),$$

where

$$\boldsymbol{b}_A := \{b_e \mid e \notin A\} \cup \{1/b_e \mid e \in A\}.$$

Proof. The proof is similar to the proof of Proposition 7.5. We have

$$Q\left(G;\, (\boldsymbol{b}, \mathbf{1}, \mathbf{0})^{\delta(A)}, c\right) = \sum_s (\omega_Z)^{\delta(A)}(s) c^{c(s)} = \sum_s \left(\prod_{\substack{v_e\in Wh(s)\\ e\notin A}} b_e\right)\left(\prod_{\substack{v_e\notin Wh(s)\\ e\in A}} b_e\right) c^{c(s)},$$

where the sum is over all graph states s with no crossing states and where $Wh(s)$ is the set vertices with white split states in the graph state s.

We can define a bijection between the set of embedded spanning subgraphs of G and the set of graph states of G_m by associating a spanning subgraph H_s of G by setting $e \in H_s$ if and only if the vertex $v_e \in Wh(s)$. It is then clear that for every graph state $c(s) = f(H_s)$. By using this bijection, we can write the above state sum as

$$= \left(\prod_{\substack{e\in E(G)\\ e\in A}} b_e\right) \sum_{H\subseteq G} \left(\prod_{\substack{e\in E(H)\\ e\notin A}} b_e\right)\left(\prod_{\substack{e\in E(H)\\ e\in A}} \frac{1}{b_e}\right) c^{f(H)} = \left(\prod_{\substack{e\in E(G)\\ e\in A}} b_e\right) Z\left(G; 1, \boldsymbol{b}_A, c, 1\right)$$

where

$$\boldsymbol{b}_A := \{b_e \mid e \notin A\} \cup \{1/b_e \mid e \in A\}$$

as required. □

Our partial duality relation is given in Theorem 7.11, which is a a multivariable and partial duality extension, in terms of the topochromate as opposed to the topological Tutte polynomial, of the duality relation in [34, 33].

Theorem 7.11. *Let G be an embedded graph with $A \subseteq E(G)$. Then*

$$Z(G; 1, \boldsymbol{b}, c, 1) = \left(\prod_{e \in A} b_e \right) Z(G^{\delta(A)}; 1, \boldsymbol{b}_A, c, 1),$$

where

$$\boldsymbol{b} = \{b_e | e \in E(G)\} \quad \textit{and} \quad \boldsymbol{b}_A = \{b_e \mid e \notin A\} \cup \{1/b_e \mid e \in A\}.$$

Proof. We have

$$\left(\prod_{e \in A} b_e \right) Z(G^{\delta(A)}; 1, \boldsymbol{b}_A, c, 1) = Q\left(G^{\delta(A)}; (\boldsymbol{b}, \mathbf{1}, \mathbf{0})^{\delta(A)},\, c\right) = Q\left(G; (\boldsymbol{b}, \mathbf{1}, \mathbf{0}), c\right) = Z(G; 1, \boldsymbol{b}, c, 1),$$

where the first equality is by Lemma 7.10, the second is by Theorem 5.6 and the third follows from Proposition 7.5. □

The signed Bollobás-Riordan polynomial was introduced by Chmutov and Pak in [21] to extend some relations between the topological Tutte polynomial and the Jones polynomial of a virtual link, and then Vignes-Tourneret [86] provided a multivariate generalization of the signed Bollobás-Riordan polynomial. In the remainder of this section we relate the duality relation in Theorem 7.11 to Vignes-Tourneret's duality relation.

A *signed* embedded graph is an embedded graph such that each edge is equipped of a sign $+$ or $-$. We let $E_\pm(G)$ denote the set of $\pm$ signed edges of G, and $e_\pm(G) = |E_\pm(G)$. Also, if H is an embedded spanning subgraph of G, then $\bar{H}$ is defined to be the complementary embedded spanning subgraph $\bar{H} = (V(G), E(G) - E(H))$, and $s(H) = \frac{1}{2}(e_-(H) - e_-(\bar{H})$.

If G is a signed embedded graph and $A \subseteq E(G)$, then the partial dual $G^{\delta(A)}$ should also be regarded as a signed embedded graph. Signs are assigned to the edges of $G^{\delta(A)}$ using the signs of the edges of G in the following way: if $e \notin A$, then retain the sign of the edge e; and if $e \in A$, then change the sign of the edge e.

Definition 7.12. The *multivariate signed Bollobás-Riordan polynomial* is

$$\tilde{Z}(G; q, \boldsymbol{\alpha}, c) = \sum_{H \subseteq G} q^{k(H)+s(H)} \left(\prod_{\substack{e \in E_+(H) \\ \cup E_-(\bar{H})}} \alpha_e \right) c^{f(H)}.$$

Various properties of the multivariate signed Bollobás-Riordan polynomial were shown by Vignes-Tourneret [86] including its invariance under partial duality of signed embedded graphs. Below we will show that the multivariate signed Bollobás-Riordan polynomial $\tilde{Z}$ is a reformulation of the topochromatic polynomial Z and that the partial duality invariance of $\tilde{Z}$ follows from Theorem 7.11. We also note that conversely, Theorem 7.11 follows from the partial duality relation for $\tilde{Z}$ given in [86].

The polynomials Z and $\tilde{Z}$ are actually equivalent up to a prefactor.

Lemma 7.13. *Let G be an embedded graph, then*

$$\tilde{Z}(G; q, \boldsymbol{\alpha}, c) = \left(\prod_{E_-(G)} q^{-1/2} \alpha_e \right) Z(G; q, \boldsymbol{\beta}, c, 1)$$

where $\boldsymbol{\beta} = \{\alpha_e \mid e \in E_+(G)\} \cup \{q\alpha_e^{-1} \mid e \in E_-(G)\}$.

The proof of the lemma is very similar to the rewriting of the signed Ribbon graph polynomial in Section 4.1 of [67].

Proof. We have

$$s(H) = \frac{1}{2}(e_-(H) - e_-(\bar{H})) = \frac{1}{2}(e_-(H) - e(G) + e_+(G) + e_-(H)) = (e_-(H) - \frac{1}{2}e_-(G)),$$

so

$$q^{s(H)} = q^{(e_-(H)-\frac{1}{2}e_-(G))}. \tag{7.1}$$

Also $E_-(\bar{H}) = E_-(G) - E_-(H)$, giving

$$\prod_{e\in E_-(\bar{H})} \alpha_e = \left(\prod_{e\in E_-(G)} \alpha_e\right)\left(\prod_{e\in E_-(H)} \alpha_e^{-1}\right). \tag{7.2}$$

Therefore

$$\begin{aligned}\tilde{Z}(G; q, \boldsymbol{\alpha}, c) &= \sum_{H\subseteq G} q^{k(H)+s(H)} \left(\prod_{e\in E_+(H)} \alpha_e\right)\left(\prod_{e\in E_-(\bar{H})} \alpha_e\right) c^{f(H)} \\ &= \sum_{H\subseteq G} q^{k(H)+s(H)} \left(\prod_{e\in E_+(H)} \alpha_e\right)\left(\prod_{e\in E_-(G)} q^{-1/2}\alpha_e\right)\left(\prod_{e\in E_-(H)} q\alpha_e^{-1}\right) c^{f(H)},\end{aligned}$$

where the second equality follows by equations (7.1) and (7.2).

Finally, the state sum above is just $\left(\prod_{e\in E_-(G)} q^{-1/2}\alpha_e\right) Z\left(G; q, \boldsymbol{\beta}, c, 1\right)$ where $\beta_e = \alpha_e$ if $e \in E_+(G)$ and $\beta_e = q\alpha_e^{-1}$ if $e \in E_-(G)$. □

We now prove the partial duality result of Vignes-Tourneret [86] as a corollary of Theorem 7.11.

Corollary 7.14. *Let G be an embedded graph. Then*

$$\tilde{Z}(G; 1, \boldsymbol{\alpha}, c) = \tilde{Z}(G^{\delta(A)}; 1, \boldsymbol{\beta}, c)$$

Proof of Corollary 7.14. By Lemma 7.13 we have

$$\tilde{Z}\left(G; 1, \boldsymbol{\alpha}, c\right) = \left(\prod_{e\in E_-(G)} \alpha_e\right) Z\left(G; 1, \beta, c, 1\right), \tag{7.3}$$

where

$$\boldsymbol{\beta} = \{\alpha_e \mid e \in E_+(G)\} \cup \{\alpha_e^{-1} \mid e \in E_-(G)\}.$$

Theorem 7.11 then gives

$$Z\left(G; 1, \beta, c, 1\right) = \left(\prod_{e\in A} \beta_e\right) Z\left(G^{\delta(A)}; 1, \boldsymbol{\beta}_A, c, 1\right), \tag{7.4}$$

where

$\boldsymbol{\beta}_A = \{\alpha_e \mid e \in E_+(G),\ e \notin A\} \cup \{\alpha_e^{-1} \mid e \in E_-(G),\ e \notin A\} \cup \{\alpha_e^{-1} \mid e \in E_+(G),\ e \in A\} \cup \{\alpha_e \mid e \in E_-(G),\ e \in A\}.$

Using the facts that $E_\pm(G)\cap A = E_\mp(G^{\delta(A)})\cap A$ and $E_\pm(G)\cap\bar{A} = E_\pm(G^{\delta(A)})\cap\bar{A}$, where $\bar{A}$ is the complement of A in $E(G)$, we can rewrite $\boldsymbol{\beta}_A$ as

$$\boldsymbol{\beta}_A = \{\alpha_e \mid e \in E_+(G^{\delta(A)})\} \cup \{\alpha_e^{-1} \mid e \in E_-(G^{\delta(A)})\}.$$

Using this rewriting of $\boldsymbol{\beta}$, Lemma 7.13 gives

$$\tilde{Z}(G^{\delta(A)}; 1, \boldsymbol{\alpha}, c) = \left(\prod_{e\in E_-(G^{\delta(A)})} \alpha_e\right) Z\left(G^{\delta(A)}; 1, \boldsymbol{\beta}_A, c, 1\right).$$

Together with Equations (7.3) and (7.4), the above identity tells us that

$$\tilde{Z}\left(G; 1, \boldsymbol{\alpha}, c\right) = \left(\prod_{e\in E_-(G)} \alpha_e\right)\left(\prod_{e\in A} \beta_e\right)\left(\prod_{e\in E_-(G^{\delta(A)})} \alpha_e^{-1}\right) \tilde{Z}(G^{\delta(A)}; 1, \boldsymbol{\alpha}, c) \tag{7.5}$$

To prove the corollary, it remains to show that the factor on the right hand side of (7.5) is 1.

$$\left(\prod_{e\in E_-(G)}\alpha_e\right)\left(\prod_{e\in A}\beta_e\right)=\left(\prod_{\substack{e\in E_-(G)\\ e\in A}}\alpha_e\right)\left(\prod_{\substack{e\in E_-(G)\\ e\notin A}}\alpha_e\right)\left(\prod_{\substack{e\in E_+(G)\\ e\in A}}\alpha_e\right)\left(\prod_{\substack{e\in E_-(G)\\ e\in A}}\alpha_e^{-1}\right)$$

$$=\left(\prod_{\substack{e\in E_-(G)\\ e\notin A}}\alpha_e\right)\left(\prod_{\substack{e\in E_+(G)\\ e\in A}}\alpha_e\right)=\left(\prod_{\substack{e\in E_-(G^{\delta(A)})\\ e\notin A}}\alpha_e\right)\left(\prod_{\substack{e\in E_-(G^{\delta(A)})\\ e\in A}}\alpha_e\right)=\prod_{e\in E_-(G^{\delta(A)})}\alpha_e.$$

The result then follows from Equation (7.5). □

We note that further analogues of the properties of the polynomial $\tilde{Z}$ which were shown in [86], can be deduced for Z, but do not include the details here.

8. Further directions

Although the theory of twisted duality has answered many questions, many more have arisen in the course of this investigation. We consolidate here some questions mentioned in previous sections, as well as a few others.

(i) We have seen that twisted duality is intimately related to medial graphs and checkerboard colourability, but further depths remain. For example, if G is any embedded 4-regular graph, which of its twisted duals are also 4-regular and checkerboard colorable? These would be the twisted duals that are actually the embedded medial graphs of some embedded graph. Suppose G and H are non-isomorphic twisted duals that are both checkerboard colorable. How are the cycle family graphs of G and H related? Also, is it possible to characterize those embedded graphs, without degree restrictions, that have a checkerboard colorable twisted dual? Clearly every bipartite graph does (take its full dual), and this question is equivalent to characterizing those G such that $Orb(G)$ contains a bipartite graph.

(ii) In this paper we investigated the orbits of the ribbon group action, in particular determining the the orbits of some special subgroups. However, there are other subgroups of the ribbon group whose actions would be worth exploring. Perhaps more importantly, it remains to understand various stabilizer subgroups of the ribbon group action. Is it possible to characterize embedded graphs that are "self twisted dual", that is, to determine when, for some $\Gamma=\prod_{i=1}^{6}\xi_i(A_i)$, G^{Γ} is equivalent to G as embedded graphs, combinatorial maps or abstract graphs.

(iii) A fundamental question that we have not addressed here is how topological invariants of an embedded graph G, such as orientability, genus, degree sequence, or number of vertices, vary over the elements in an orbit of the ribbon group action. Is it possible to determine at least ranges for any of these invariants? More ambitiously, is it possible to determine any of these invariants for $G^{\boldsymbol{\xi}}$ just from G and $\boldsymbol{\xi}$?

(iv) Although we focused mainly on topological graph polynomials here, questions also abound for classical graph polynomials such as the Tutte or chromatic polynomial (see [31, 32] for a survey of graph polynomials). In particular, how are various graph polynomials constrained on $Orb(G)$ for a fixed G?

(v) The Penrose polynomial has the following properties for a plane graph G: the coefficients of $P(G;\lambda)$ alternate in sign, $P(G;3)$ counts the number of edge 3-colourings, and G has bridge if and only if $P(G;\lambda)=0$. To what extent do these properties hold in the non-plane case?

(vi) Theorem 6.9, which is due to Jaeger, states that for each plane graph G, the Penrose polynomial $P(G;k)$ is equal to the number of admissible k-valuations of the medial ribbon graph G_m of G. Can this result be extended to: if G is the twisted dual of a plane graph, then for each $k\in\mathbb{N}$, $P(G;k)$ is equal to the number of admissible k-valuations of the medial ribbon graph G_m of G. The characterization of the partial duals of plane graphs from [69] may apply to this problem.

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